CO-ORDINATE GEOMETRY

FOR

B. A. and B. Sc. STUDENTS

BY

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CONTENTS.

CHAPTER I.

ELEMENTARY.

CHAPTER II.

THE STRAIGHT LINE.

CHAPTER III.

THE CIRCLE.

PREFACE.

Almost all the standard books on Co-ordinate Geometry available in the market have been designed, not to meet the requirements of any particular class of students, but to deal with the subject more or less exhaustively. The result is that whereas a candidate going up for his Intermediate Examination, has to omit large portions of these books here and there which naturally detracts from his satisfaction of having mastered the book, the candidate going up for his degree examination has to wade through a mass of elementary and unimportant matter which is an avoidable encroachment upon his precious time.

To meet the needs of the students preparing for the Intermediate Examination, we brought out our Co-ordinate Geometry for Intermediate Classes, whose second edition is now in the press. The present volume is a natural sequel to it and is designed to suit the needs of B.A. and B.Sc. students.

Some of the special features of the book are :—

1. Every attempt has been made to see that it covers the course of B.A., and B.Sc., students in every Indian University.

2. Propositions which are too simple and elementary, to be treated in detail in an advanced treatise, have been

taken for granted, except for a statement of their results for ready reference.

3. Propositions at which the students generally fumble have been carefully explained and suitably illustrated.

4. The number of examples is large enough to give a good grounding of the subject, but not so disproportionately large as to scare away the reader. Almost every important book article has a few attached examples to afford practice in its application.

As in the case of our earlier book, no originality is claimed for this volume either. All books available on the subject have been freely consulted and drawn upon, and our grateful thanks are due to their authors and publishers. All that we claim is the presentation of the subject in a handier form for the students of B.A. and B.Sc., classes, based on an experience of teaching extending over more than two decades.

We hope the book will serve the purpose for which it is intended.

Meerut.
1-7-'42.

Madan Mohan
Kailash Prakash

CHAPTER IV.

CONICS.

CHAPTER V.

PARABOLA.

CHAPTER VI.

ELLIPSE.

CHAPTER VII.

HYPERBOLA.

CHAPTER VIII.

POLAR CO-ORDINATES.

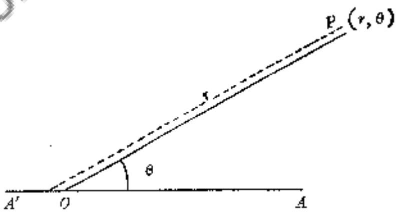
CHAPTER I

ELEMENTARY

1·1. In the Cartesian System of Co-ordinates, the position of a point in a plane can be definitely fixed, if its distances along two fixed intersecting straight lines are known. There can be many other ways of fixing the position of a point. We propose to discuss here one of these *viz.* *The Polar System of Co-ordinates*.

1·2. *Given a line in a plane and a point on this line, the position of any point in this plane can be fixed, if (i) its distance from the given point, and (ii) the angle which the line joining it to the given point makes with the given line, are known.*

Let $A'A$ be the given line, and O the given point on it. The position of any point P is known if OP and the $\angle POA$ are known. P, we may say, is the point $(OP, \angle POA.)$



The fixed line is known as the *Initial line,* and the fixed point as the *Pole*. OP is known as the *Radius Vector* or the *r—co-ordinate* and is generally denoted by r. Angle POA is known as the *Vectorial Angle* or the *θ—co-ordinate* and is generally denoted by θ.

The radius vector is *positive,* if it be measured from the pole in the same direction as the boundary line of the vectorial angle, and *negative* if measured in the opposite direction.

The vectorial angle is *positive,* if it be measured from the initial line in the anti-clockwise direction and *negative,* if measured in the clockwise direction.

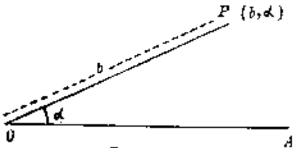


Figure I

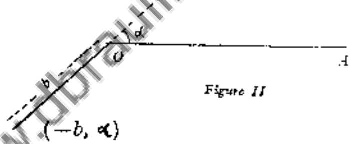


Figure II

Thus in *Fig. I,* b and α both are positive; in *Fig. II,* b is negative and α is positive; in *Fig. III,* b and α are both negative; whereas in *Fig. IV,* b is positive and α is negative.

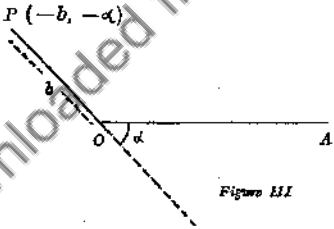


Figure III

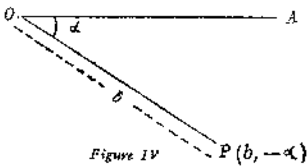


Figure IV

The system where the point is so defined is known as the *Polar System of Co-ordinates.*

It is clear from the above that the same point is denoted by each of the four sets of polar co-ordinates,

(r, θ), $(-r, \pi+\theta)$, $\{r, -(2\pi-\theta)\}$, $\{-r, -(\pi-\theta)\}$.

It is also clear that adding 2π or any multiple of 2π to the vectorial angle of any point does not alter the position of that point.

Ex. 1. Determine the positions of the points

$(3,30°)$, $(-3,30°)$, $(-3, -30°)$ and $(3, -30°)$

and find the area of the figure obtained by joining them.

Ex. 2. Where do all points lie whose

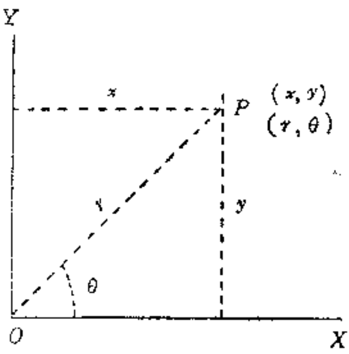
(a) Vectorial angles are
(i) Zero, (ii) 30° and (iii) any constant.

(b) Radii vectores are
(i) 5, (ii) -3 and (iii) any constant.

1·3. *To establish a relation between the Cartesians and the Polars.*

Case I. *When the pole coincides with the origin, and the initial line lies along one of the axes of co-ordinates, say the axis of x.*

Let (x, y) be the cartesian and (r, θ) the polar co-ordinates of the point P.



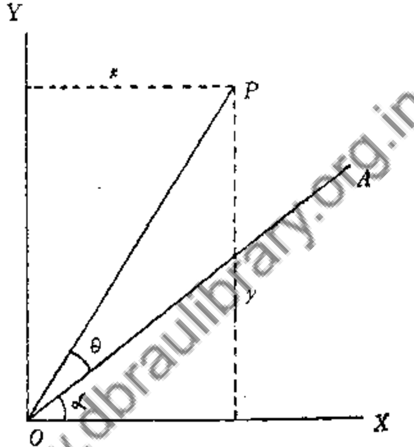
It is quite evident from the figure, that

$$\left.\begin{aligned} x &= r\cos\theta \\ \text{and } y &= r\sin\theta \end{aligned}\right\} \quad \ldots\ldots\ldots (i)$$

First squaring and adding, and then dividing one by the other, we get

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \text{and } \theta &= \tan^{-1} y/x \end{aligned}\right\} \quad \ldots\ldots (ii)$$

Case II. *When the pole coincides with the origin and the initial line is inclined at any angle α to the axis of x.*



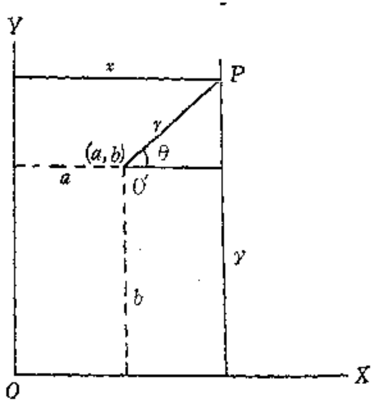
In this case

$$\left.\begin{array}{l} x=r\cos(\theta+\alpha) \\ y=r\sin(\theta+\alpha) \end{array}\right\}\ldots\ldots(i)$$

And, as before

$$\left.\begin{array}{l} r^2=x^2+y^2 \\ \theta=\tan^{-1}\dfrac{y}{x}-\alpha \end{array}\right\}\ldots(ii)$$

Case III. *When the pole is situated at the point (a, b) referred to the rectangular system, and the initial line is parallel to one of the axes, say the axis of x.*



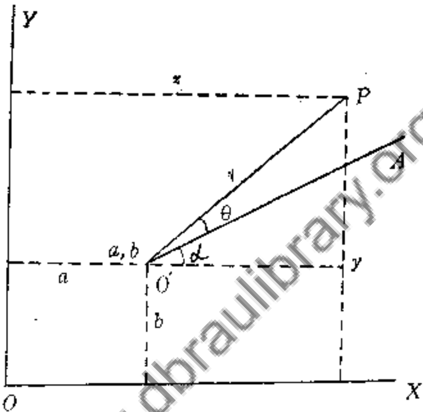
Here

$$\left.\begin{array}{l} x=r\cos\theta+a \\ y=r\sin\theta+b \end{array}\right\}\ldots\ldots(i)$$

So that

$$\left.\begin{array}{l} r^2=(x-a)^2+(y-b)^2 \\ \theta=\tan^{-1}\dfrac{y-b}{x-a} \end{array}\right\}\ldots(ii)$$

Case IV. *When the pole is situated at the point* (a, b) *referred to the rectangular system, and the initial line is inclined at any given angle* α *to one of the axes, say the axis of* x.



This is a combination of Cases *II* and *III*. It can be easily seen that here

$$\left.\begin{aligned} x &= r\cos(\theta+\alpha)+a \\ y &= r\sin(\theta+\alpha)+b \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

and
$$\left.\begin{aligned} r^2 &= (x-a)^2+(y-b)^2 \\ \theta &= \tan^{-1}\frac{y-b}{x-a}-\alpha \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Equations (i) in all these cases give the *Cartesians* in terms of the *Polars* and equations (ii) give *Polars* in terms of the *Cartesians.*

In the above figures the point P is seen to be in the first quadrant. This should not lead the student to suppose that the results arrived at are true only for positions of the point in the first quadrant. He may proceed with his point in any other quadrant, and he will see that the same results hold good.

Case I is by far the most important and its application is frequently required. Others are of rare occurrence. Whenever a change from one system to the other is suggested, unless anything is stated to the contrary, transformation of Case I should be presumed.

Ex. 1. Find the polar co-ordinates of the points whose rectangular co-ordinates are

$$(3, 0);\ (-5, -6);\ (-5, 0);\ (0, -4);\ (-2, 1)$$

Ex. 2. Find the rectangular co-ordinates of the points whose polar co-ordinates are

$$(8, 60°);\ (5, 30°);\ (9, 135°);\ (2, -60°);\ (-1, 45°) \text{ and } (-4, -30°)$$

Ex. 3. Reduce to the polar form :—

(*i*) $y = mx + c$ (*ii*) $x^2 + y^2 = a^2$

(*iii*) $x^2/a^2 + y^2/b^2 = 1$ (*iv*) $y^2 = 4ax$

(*v*) $ay^2 = x^2(x-a)$ (*vi*) $x^2(x^2+y^2) = a^2(y^2-x^2)$

(*vii*) $x^3 = y^2(2a-x)$ (*viii*) $y^3 = (x-a)^2(x-b)$.

Ex. 4. Reduce to the cartesian form :—

(*i*) $r \cos(\theta - \lambda) = p$ (*ii*) $r = 2a \cos \theta$

(*iii*) $r = a \sin \theta \cos \theta$ (*iv*) $r = a \cos \theta + b$

(*v*) $r = a (\sin \theta + \cos \theta)$ (*vi*) $r^2 = a^2 \cos 2\theta$

(*vii*) $r \cos \theta = a$ (*viii*) $r \sin \theta/2 = a$

1·40. *The distance between two points whose co-ordinates are given is*

(*a*) $$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

if the points are (x_1, y_1) *and* (x_2, y_2).

(I. C. G. 2·10)

(b) $\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}$,

if the points are (r_1, θ_1) *and* (r_2, θ_2)

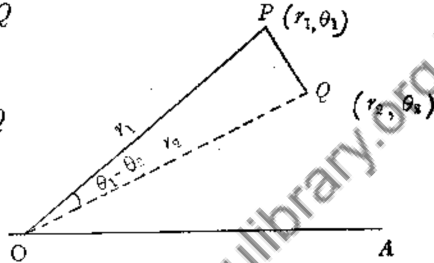
For in the triangle OPQ

$PQ^2=OP^2+OQ^2$

$-2OP.OQ\cos POQ$

$=r_1^2+r_2^2$

$-2r_1r_2\cos(\theta_1-\theta_2)$



Hence

$PQ=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}$

1·5. The co-ordinates of the point $P(x, y)$ which divides the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the ratio $p : q$ are

$$x=\frac{px_2+qx_1}{p+q} \qquad y=\frac{py_2+qy_1}{p+q}$$

$$\text{or} \quad x=\frac{px_2-qx_1}{p-q} \qquad y=\frac{py_2-qy_1}{p-q}$$

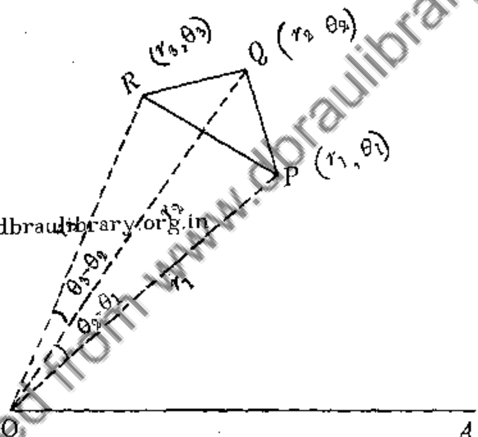
according as P is a point of internal or external division of $P_1 P_2$. (I. C. G. 2·20)

In Polars the co-ordinates of such a point are too cumbersome to be of any practical use. Hence we do not propose to discuss them here.

1·6. *The area of the triangle whose angular points are given is* (*a*) $\frac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$, *if the angular points are* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3).

(I. C. G. 2·30)

(*b*) $\frac{1}{2}\{r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)\}$, *if the angular points are* $P\ (r_1, \theta_1)$, $Q\ (r_2, \theta_2)$ *and* $R(r_3, \theta_3)$.



For

$$\triangle PQR = \triangle OPQ + \triangle OQR - \triangle OPR$$

$$=\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)+\tfrac{1}{2}r_2r_3\sin(\theta_3-\theta_2)-\tfrac{1}{2}r_1r_3\sin(\theta_3-\theta_1)$$

$$=\tfrac{1}{2}\{r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)\}$$

The above expressions for the area will lead to a positive result, if the angular points are so taken that in passing round them in the order *P, Q, R,* the area of the triangle lies to the left. Failure to observe this will lead to a negative result.

1·7. *Changes in axes are related to changes in the equations as follows :—*

Change in axes		*Change in equation*	
Origin	***Direction***	x	y
Moves to (h, k)	No change	$x+h$	$y+k$
No change	Turns through θ	$x \cos\theta - y \sin\theta$	$x \sin\theta + y \cos\theta$
Moves to (h, k)	Turns through θ	$x \cos\theta - y \sin\theta + h$	$x \sin\theta + y \cos\theta + k$

(I. C. G. 3·40)

CHAPTER II

THE STRAIGHT LINE

2·10. The equation to the straight line cutting off intercepts a and b from the axes of x and y is

$$x/a+y/b=1. \qquad \text{(I. C. G. 4·20.)}$$

2·11. The equation to a straight line inclined at an angle θ to the axis of x, and cutting off an intercept c on the axis of y is

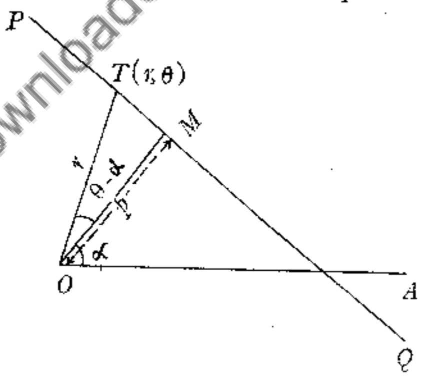
$$y=mx+c,$$

where $m=\tan\theta$. (I. C. G. 4·21.)

2·12. (*a*) The equation to a straight line on which the perpendicular from the origin is of length p, and is inclined at an angle α to the axis of x, is

$$x\cos\alpha+y\sin\alpha=p \qquad \text{(I. C. G. 4·22.)}$$

(*b*) *To find the polar equation to a straight line on which the perpendicular from the pole is of length p, and is inclined at an angle α to the initial line.*



Let (r, θ) be the co-ordinates of any point T on the given line PQ. Then

$$OT\cos(\theta-\alpha)=p$$

or $\quad r\cos(\theta-\alpha)=p$

is the required equation.

The above equation could also have been deduced from that of (*a*) by substituting $r \cos \theta$ for x and $r \sin \theta$ for y.

Ex. 1. Find the polar equation of a straight line

(*i*) at right angles to the initial line and at a distance a from the origin.

(*ii*) parallel to the initial line and at a distance a from the origin.

Ex. 2. What is the equation of a straight line perpendicular to

$$p = r \sin (\theta + \alpha)$$

and at the same perpendicular distance from the origin.

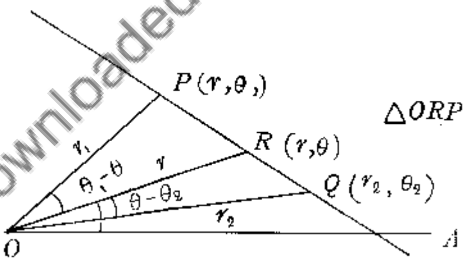
Ex. 3. A line is drawn through the point (r_1, θ_1) perpendicular to the line $\theta = \alpha + \pi/2$, shew that its polar equation is

$$r \sin (\theta - \alpha) = r_1 \sin (\theta_1 - \alpha).$$

2·13. (*a*) The equation to the straight line passing through the points (x', y') and (x'', y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x') \qquad \text{(I. C. G. 4·5)}$$

(*b*) *To find the polar equation to a straight line passing through the points P (r_1, θ_1) and Q (r_2, θ_2).*



Let $R(r, \theta)$ be any other point on this line. Then

$$\triangle ORP + \triangle OQR = \triangle OQP$$

or $\quad rr_1 \sin(\theta_1 - \theta) + rr_2 \sin (\theta - \theta_2) = r_1 r_2 \sin (\theta_1 - \theta_2)$

[Art. 1·6]

$$\text{or} \quad \frac{\sin (\theta_1 - \theta_2)}{r} + \frac{\sin (\theta_2 - \theta)}{r_1} + \frac{\sin (\theta - \theta_1)}{r_2} = 0,$$

is the required equation.

This equation could also have been deduced from the previous equation by changing cartesians therein to the polars.

Ex. 1. Find the polar equation of the straight line passing through the points :—

(i) $\left(a, \dfrac{\pi}{2}\right)$ and $\left(3a, \dfrac{\pi}{6}\right)$

(ii) $(-3, 45°)$ and $(7, 105°)$

(iii) $\left(-a, \dfrac{\pi}{6}\right)$ and $\left(-2a, -\dfrac{2\pi}{3}\right)$

Ex. 2. A and B are two points whose polar co-ordinates are (r_1, θ_1) and (r_2, θ_2) respectively Find the co-ordinates of the point where the bisector of the angle AOB cuts the line AB.

2·2. The length of the perpendicular from any point $P\ (x', y')$ upon the straight line $Ax+By+C=0$ is

$$\pm \frac{Ax'+By'+C}{\sqrt{A^2+B^2}} \qquad \text{(I. C. G. 5·12)}$$

If C is assumed negative, the plus or the minus sign is taken according as P and the origin lie on opposite sides or same side of the straight line. If C is positive, these signs are reversed.

2·3. $\pm \dfrac{m_1 - m_2}{1+m_1 m_2}$ is the tangent of the angle between the straight lines $y=m_2x+c_1$ and $y=m_2x+c_2$.

(I. C. G. 5·21)

The sign preceding the above expression is positive or negative according as the angle is measured, starting from the straight line $y=m_1x+c_1$, in the counter-clockwise or the clock-wise direction.

2·4. The equations to the bisectors of the angles between the straight lines $a_1x+b_1y+c_1=o$ and $a_2x+b_2y+c_2=o$ are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{(I. C. G. 5·40)}$$

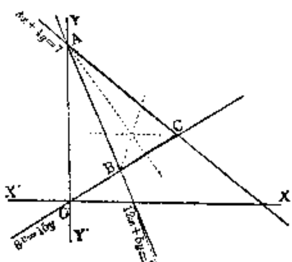
The sign here is taken as positive or negative according as the origin lies or does not lie within the angle of which the bisector is required, provided that c_1 and c_2 both are of the same sign.

If either one or both the lines pass through the origin, the above criterion fails. In the former case, the nature of the intercepts which these bisectors cut off on the axes, and in the latter the nature of the angles which they make with the axes enable us to differentiate between them, as will be illustrated by the following examples :—

Ex 1. Find the equations of the internal bisectors of the angles of the triangle formed by the lines.

$$3x+4y=7\ ;\ 12x+5y=9 \text{ and } 8x=15y.$$

Let ABC be the given triangle. The equations to the bisectors of the angles at A are



$$\frac{3x+4y-7}{5} = \pm \frac{12x+5y-9}{13}$$

As the origin does not lie within the angle BAC, and the constant term in both is of the same sign, the proper equation to the internal bisector at A is

$$\frac{3x+4y-7}{5} = -\frac{12x+5y-9}{13}$$

or $$99x+77y-136=0$$

Again the equations to the bisectors at B are

$$\frac{12x+5y-9}{13} = \pm \frac{8x-15y}{17}$$

or $\quad 100x+280y-153=0$

and $\quad 308x-110y-153=0$

In this case the origin lies on one of the lines and not in any of the angles. Therefore the criterion of Art. 2·4 fails. But we see from the figure that the internal bisector of the angle B cuts off a positive intercept on the axis of x and a negative intercept on the axis of y, whereas the external one cuts off positive intercepts on both the axes. Hence the equation to the required bisector is

$$308x-110y=153.$$

Also the equations to the bisectors at C are

$$\frac{3x+4y-7}{5} = \pm \frac{8x-15y}{17}$$

or $\quad 11x+143y-119=0$

and $\quad 91x-7y-119=0$

The internal bisector at C cuts off positive intercepts from both the axes. Hence its equation is

$$11x+143y-119=0$$

Ex 2. *Find the external bisectors of the angles of the triangle formed by the line* $6x+8y=9$ *with the axes.*

The equations to the bisectors of the angles at A

are $\quad \dfrac{6x+8y-9}{10} = \pm y$

i.e. $\quad 6x-2y-9=0$

and $6x+18y-9=0$

Those at B are

$$\frac{6x+8y-9}{10} = \pm x$$

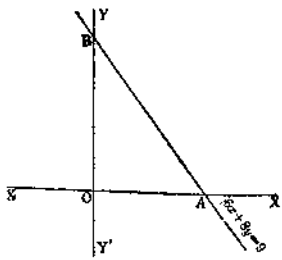
i.e. $\quad 4x-8y+9=0$

and $16x+8y-9=0$

And those at O are

$$x=\pm y$$

i.e. $\quad x-y=0$ and $x+y=0$.



By the criterion employed in the last example the external bisectors at A and B are at once seen to be

$$6x-2y-9=0 \text{ and } 4x-8y+9=0 \text{ respectively.}$$

But for discriminating between the external and internal bisectors at O even this criterion is not applicable, since both these bisectors pass through the origin and hence have no intercepts on the axes.

It is apparent from the figure, however, that for the internal bisector at O, m is positive whereas for the external bisector it is negative.* Hence the required external bisector is $x+y=0$.

Ex 3. The equations to the three sides of a triangle are $x=3$, $y=4$ and $4x+3y=12$. Find the equations to the bisectors of the three angles of the triangle.

Ex 4. Find the internal bisectors of the angles of the triangle whose sides are

$$4x+3y+7=0, \quad 5x+12y-20=0 \text{ and } 3x+4y+8=0.$$

Ex. 5. Find the internal bisectors of the angles of the triangle whose vertices are (0,0), (3,3) and (5,1). Also find the angles between these bisectors.

2·51. Two straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ intersect in

$$\left\{\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \quad \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right\} \qquad \text{(I. C. G. 5·50.)}$$

2·52. $a_1x+b_1y+c_1+k\,(a_2x+b_2y+c_2)=0$ is the equation to a straight line passing through the point of intersection of the straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. (I. C. G. 5·52.)

2·53. The condition that the straight lines given by $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ meet in a point is

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* Both m's can not have the same sign because the two bisectors being perpendicular, their product is -1.

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0$$

i.e.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

or

That three integers l, m, and n can be found such that $l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$.

(I. C. G. 5·54.)

2·6. The co-ordinates of the in-centre of the triangle whose angular points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, are

$$\frac{ax_1+bx_2+cx_3}{a+b+c}, \quad \frac{ay_1+by_2+cy_3}{a+b+c}$$

The co-ordinates of the ex-centres of the above triangle touching the sides BC, CA and AB are

$$\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \quad \frac{-ay_1+by_2+cy_3}{-a+b+c}$$

$$\frac{ax_1-bx_2+cx_3}{a-b+c}, \quad \frac{ay_1-by_2+cy_3}{a-b+c}$$

$$\frac{ax_1+bx_2-cx_3}{a+b-c}, \quad \frac{ay_1+by_2-cy_3}{a+b-c}$$

where a, b and c are the lengths of the sides of the triangle.

(I. C. G. 5·60.)

2·7. *If an equation of any degree whatever can be broken up into factors, the locus represented by this equation is composed of the loci represented by each one of these factors.*

As a simple illustration, let us consider the equation

$$x^2-y^2=0 \quad \dots \quad \dots \quad (1)$$

$$\text{or } (x-y)(x+y)=0 \quad \dots \quad (2)$$

Now (1) is satisfied by all those values of x and y which satisfy any one of the two equations

$$\left.\begin{aligned} x+y&=0 \\ \text{and}\quad x-y&=0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

That is to say, the co-ordinates of any point lying on any one of the two straight lines given by (3) satisfy (1).

Hence equation (1) represents the two straight lines given by equation (3).

Although the illustration chosen is a simple one, the method followed is general, hence the proposition.

The converse of the above proposition is obvious *viz.* that if a number of expressions in x and y equated to zero, giving a number of curves, be multiplied together, their product equated to zero, will be the combined equation of all these curves.

Ex. 1. Find the equations to the different curves represented by :—

(i) $x^3+6x^2y+11xy^2+6y^3=0$

(ii) $x^4-5x^3y+6x^2y^2-5xy^3+y^4=0$

(iii) $(1+x)^2(1+y^2)-(1+y)^2(1+x^2)=0$

Ex. 2. Find the equation of the pair of straight lines through (2,5) which make angles of 30° on each side of the straight line whose equation is

$$2y-x+1=0$$

2·81. A homogenous equation of the second degree in x and y represents two straight lines passing through the origin, real and different, coincident or imaginary* according as $h^2 >\ =$ or $< ab$. (I. C. G. 6·40.)

*The idea of imaginary straight lines has been introduced to complete the theory and to keep up the uniformity of statements in Analytical Geometry. As a matter of fact the same equation which represents a pair of *imaginary* straight lines, *in reality* represents only a point, the point of intersection of these imaginary straight lines.

2·82. If θ be the angle between the lines given by $ax^2+2hxy+by^2=0$, then

$$\tan\theta = 2\sqrt{h^2-ab}/(a+b).$$

These straight lines are parallel if $h^2=ab$, and at right angles if $a+b=0$. (I. C. G. 6·41.)

2·83. The equation to the bisectors of the angles between the straight lines given by $ax^2+2hxy+by^2=0$

is $$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$ (I. C. G. 6·42.)

2·84. *To find the condition that the general equation of the second degree i.e., the equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

(where a, b, c, f, g and h are any constants) represents a pair of straight lines.

If any one equation represents two straight lines, by art. 2·7 it must be resolvable into two linear factors. Therefore the problem before us reduces to finding the condition that the given equation may be so resolvable.

Writing the given equation as a quadratic in x, we have

$$ax^2+2x(hy+g)+(by^2+2fy+c)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

or $\{a^2x^2+2ax(hy+g)+(hy+g)^2\}=(hy+g)^2-a(by^2+2fy+c)$

or $(ax+hy+g)^2=y^2(h^2-ab)+2y(gh-af)+g^2-ac \ldots\ldots\ldots(2)$

In order that (2) may be resolvable into two linear factors the expression on the right must be a perfect square

i.e. $(gh-af)^2=(h^2-ab)(g^2-ac)$

or $g^2h^2-2afgh+a^2f^2=h^2g^2-abg^2-ach^2+a^2bc$

or $abc+2fgh-af^2-bg^2-ch^2=0. \quad \ldots \quad \ldots \quad \ldots \quad (3)$

The above expression is called the discriminant of the general equation of the second degree, and is generally denoted by $\triangle$.

The student acquainted with determinant notation will see that

$$\triangle = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Ex. 1. Show that the equation

$$6x^2+17xy+12y^2+22x+31y+20=0$$

represents a pair of straight lines. Find their equations.

Also find the equation of the straight lines passing through the origin and

(*i*) Parallel to them

(*ii*) Perpendicular to them

Ex. 2. Prove that the equation

$$a^2(x \cos\theta - y \sin\theta)^2 = x^2+y^2-2ax \sin\theta - 2ay \cos\theta + a^2$$

represents a pair of straight lines, and show that the perpendicular distance from the origin to either of the lines is unity.

Ex 3. Show that the equations

$$3x^2+4xy-4y^2-14x+12y-5=0$$

and $$3x^2-11xy+6y^2+22x-17y+7=0$$

represent pairs of straight lines, having one line in common.

2·85. *Given that the general equation of the second degree represents a pair of straight lines; to find their point of intersection.*

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$.................(1) be the general equation of the second degree, and let (x_1, y_1) be the point of intersection of the straight lines represented by it.

On transferring the origin to (x_1, y_1), it becomes

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2$$
$$+2g(x+x_1)+2(y+y_1)+c=0$$

or $$ax^2+2hxy+by^2+2x(ax_1+hy_1+g)$$
$$+2y(hx_1+by_1+f)+(ax_1^2+2hx_1y_1$$
$$+by_1^2+2gx_1+2fy_1+c)=0 \quad \ldots\ldots(2)$$

As (2) now represents a pair of straight lines passing through the origin, it must be a homogenous equation of the second degree in x and y; *i.e.*, the co-efficients of x and y therein, as also the constant term must simultaneously be zeros.

Whence $$ax_1+hy_1+g=0 \quad \ldots\ldots (3)$$
$$hx_1+by_1+f=0 \quad \ldots\ldots (4)$$

and $$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \ldots\ldots (5)$$

From (3) and (4)

$$\left.\begin{aligned} x_1 &= \frac{hf-bg}{ab-h^2} \\ y_1 &= \frac{gh-af}{ab-h^2} \end{aligned}\right\} \quad \ldots\ldots (6)$$

Hence $\left(\dfrac{bg-hf}{h^2-ab}, \dfrac{af-gh}{h^2-ab}\right)$ is the required point.

Cor. Equation to the bisectors of (1) is

$$\frac{(x-x_1)^2-(y-y_1)^2}{a-b} = \frac{(x-x_1)(y-y_1)}{h} \qquad \text{(Art. 2·83).}$$

Ex 1. Find the distance of the point of intersection of the straight lines $6x^2-xy-12y^2+15x+20y=0$ from the line $y=2x+1$.

Ex 2. Find the co-ordinates of the foot of the perpendicular from the point of intersection of $2x^2-3xy-2y^2+5x+5y-3=0$ to the straight line $5x-7y=4$.

Ex 3. If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two straight lines, prove that the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2}$$

2·86. On the assumption that the general equation of the second degree represents a pair of straight lines, equation (5) of the previous article must be true;

i.e., $$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$$

or $$x_1(ax_1+hy_1+g)+y_1(hx_1+by_1+f)+(gx_1+fy_1+c)=0$$

or $$gx_1+fy_1+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

[From (3) and (4)]

Substituting for x_1 and y_1 from (6) in (7), we have

$$g(bg-hf)+f(af-gh)+c(h^2-ab)=0$$

or $$abc+2fgh-af^2-bg^2-ch^2=0$$

This method can be looked upon as an alternative method for finding the above condition.

2·87. *Given that the general equation of the second degree represents a pair of straight lines; to find the angle between the pair.*

Let $$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots\ldots(1)$$

be the general equation of the second degree, and let $y=m_1x+c_1$ and $y=m_2x+c_2$ be the equations to the two lines represented by it.

Then

$$ax^2+2hxy+by^2+2gx+2fy+c$$
$$=b(y-m_1x-c_1)(y-m_2x-c_2)=0 \quad \ldots\ldots(2)$$

From (2)

$$\left.\begin{aligned} m_1+m_2 &= -2h/b \\ m_1m_2 &= a/b \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\left.\begin{aligned} c_1+c_2 &= -2f/b \\ c_1c_2 &= c/b \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$m_1c_2+m_2c_1 = 2g/b\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now if θ be the required angle,

$$\tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}$$

$$= \pm\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$= \pm\frac{2\sqrt{h^2-ab}}{a+b}\quad\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Thus if $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, the angle between this pair of straight lines is the same as the angle between the pair represented by the homogenous equation $ax^2+2hxy+by^2=0$.

In fact since the values of m_1 and m_2 above depend only on a, h and b (the co-efficients of the second degree terms in the equation), if there be two second degree equations, each representing a pair of straight lines and identical in their second degree terms, the straight lines given by one of these equations are parallel, each to each, to the straight lines given by the other equation.

Ex. 1. What must be the values of k and p, so that the equation

$$2x^2+kxy+y^2+2x-3y+p=0$$

may represent a pair of straight lines parallel to the pair

$$2x^2+3xy+y^2=0.$$

Ex. 2. Find the equation to the pair of straight lines intersecting at (1, 2) and

(*i*) parallel

and (*ii*) perpendicular

to the pair $4x^2+17xy+15y^2=0$

Ex. 3. Find the equation to the pair of straight lines, at right angles to those represented by $x^2+xy-6y^2+7x+31y-18=0$ and passing through their point of intersection

2·9. *To find the equation to the straight lines joining the origin to the points in which the straight line*

$$lx+my=n \quad \dots\dots(1)$$

intersects the curve given by

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \dots\dots(2)$$

Consider the equation

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{lx+my}{n}\right)$$

$$+c\left(\frac{lx+my}{n}\right)^2=0 \quad \dots\dots(3)$$

Values of x and y that satisfy equations (1) and (2), necessarily satisfy (3); *i.e.*, the points that lie on the straight line (1) and the curve (2), also lie on the curve given by (3). In other words points common to (1) and (2) lie on (3), or the curve given by (3) passes through the common points of (1) and (2).

Also (3) being a homogenous equation of the second degree represents a pair of straight lines passing through the origin.

Therefore (3) is the required equation.

A handy working rule may be laid down thus :—

Make the equation to the curve homogenous with the help of the equation to the straight line. The resulting equation represents the two straight lines joining the points of intersection of the given curve and the given straight line, to the origin.

Ex 1. Prove that, if the circle

$$x^2+y^2+2gx+2fy+c=0$$

intercepts on the line $lx+my=n$ a length which subtends a right angle at the origin, then

$$c(l^2+m^2)+2n(gl+fm)+n^2=0$$

Ex. 2. Prove that the straight lines joining the origin to the points of intersection of the line $y=x-2$ and the curve $5x^2+12xy-8y^2+8x-4y+12=0$ make equal angles with the axes.

Ex. 3. Find the condition that the straight lines joining the origin to the points of intersection of the line $y=mx+c$ and the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be coincident

Examples.

1. Show that the equation $\lambda y^2+(1-\lambda^2)xy-\lambda x^2=0$ where λ is a constant, represents a pair of perpendicular straight lines. Obtain the equation referred to these as the axes of co-ordinates, of the straight lines whose equation referred to the original axes is $x(1-\lambda)+y(1+\lambda)=\sqrt{1+\lambda^2}$.

2. Show that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is

$$\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}.$$ [*Agra Uni.* 1934]

3. Prove that the product of the perpendiculars from the origin on the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{is} \quad \frac{c}{\sqrt{4h^2+(a-b)^2}}$$

4. Find the distance between the pair of parallel lines represented by

$$x^2+2\sqrt{3}xy+3y^2-3x-3\sqrt{3}y-4=0$$

5. Assuming that the general equation of the second degree represents two parallel straight lines, shew that

(i) $af^2=bg^2$

(ii) the distance between them is $2\sqrt{\dfrac{g^2-ac}{a(a+b)}}$

6. The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines at right angles to one another. Prove that the square of the distance of their point of intersection from the origin is

$$f^2+g^2\Big/b^2+h^2$$ [Agra Uni. 1929]

7. If $u\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points where these meet the axes is

$$cu+4(fg-ch)xy=0$$ [Agra Uni. 1932]

8. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines equidistant from the origin, prove that

$$f^4-g^4=c\,(bf^2-ag^2).$$ [Agra Uni. 1933]

9. Show that the ortho-centre of the triangle formed by the straight lines given by $ax^2+2hxy+by^2=0$ and $lx+my=1$ is a point (x', y') such that

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2-2hlm+bm^2}$$

10. A triangle has the lines $ax^2+2\,hxy+by^2=0$ for its sides and the point (c, d) for its ortho-centre. Prove that the third side is $(a+b)\ (cx+dy)=ad^2-2hcd+bc^2$.

11. Show that the distance of the origin from the ortho-centre of the triangle formed by the lines $\dfrac{x}{l}+\dfrac{y}{m}=1$ and $ax^2+2hxy+by=20$ is $\dfrac{(a+b)\ lm\ (l^2+m^2)^{\frac{1}{2}}}{al^2-2hlm+bm^2}$

12. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines intersecting at (x', y'), the equation of the lines bisecting the angles between them will be

$$\frac{x-x'}{ax+hy+g}=\frac{y-y'}{hx+by+f}$$

13. Shew that the straight lines joining the origin to the points of intersection of the curves

$$ax^2+2hxy+by^2+2gx=0$$
$$\text{and}\quad a'x^2+2h'xy+b'y^2+2g'x=0$$

will be at right angles, if

$g\ (a'+b')-g'\ (a+b)=0$ [*Agra Uni. 1928*]

14. If the intercept on the straight line $lx+my=1$ by the curve $x^2+y^2=a^2$ subtends an angle of 45° at the origin, shew that

$$4\{a^2\ (l^2+m^2)-1\}=\{a^2\ (l^2+m^2)-2\}^2$$

15. Show that two of the straight lines represented by the equation $ax^3+bx^2y+cxy^2+dy^3=0$ will be at right angles if $a^2+ac+bd+d^2=0$ [*Cal. Uni. 1927*]

16. Show that the equation

$$a(x^4+y^4)-4bxy\ (x^2-y^2)+6cx^2y^2=0$$

represents two pairs at right angles, and that the two pairs will coincide, if

$2b^2=a\ (a+3c)$. [*Alld. Uni. 1932*].

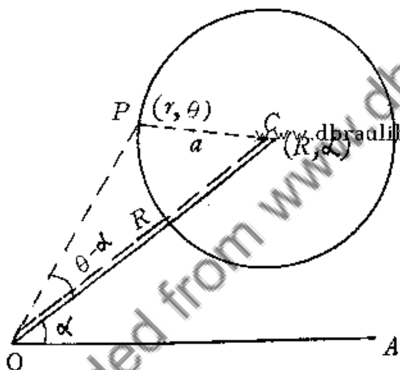
CHAPTER III.

THE CIRCLE

3.1. (*a*) The equation to a circle having any point (h, k) for its centre, and any length a for its radius is

$$(x-h)^2+(y-k)^2=a^2 \qquad \text{(I. C. G. 7·20)}$$

(b) *To find the polar equation to a circle having the point (R, α) for its centre, and any length a for its radius.*



Take any point P on the circle and join OP.

Then

$$OP^2-2OP.OC.\cos POC+OC^2=CP^2$$

or

$$r^2-2rR\cos(\theta-\alpha)+R^2=a^2$$

is the required equation.

The quadratic nature of the above equation in r indicates that OP will meet the circle in one more point.

The equation found above is the general equation to a circle in polar co-ordinates. For particular cases it will be correspondingly simplified ; *e.g.*

Circle	*Equation.*
(i) Pole as the centre	$r=a.$
(ii) Pole on the circumference, and the initial line passing through the centre	$r=2a\cos\theta.$

Circle	*Equation.*
(iii) Pole on the circumference, and the initial line inclined at an angle α to the line joining the pole and the centre	$r = 2a \cos(\theta - \alpha)$.

Ex. 1. Find the equation to a circle having the centre on the initial line and

(i) touching a straight line through the pole inclined at 45° to the initial line.

(ii) touching a straight line parallel to the initial line.

Ex. 2. Find the polar equation to a circle, the initial line being a tangent. What does it become if the origin be on the circumference ?

Ex. 3. Prove that the equation to the circle described on the straight line joining $(1, 60°)$ and $(2, 30°)$ as diameter is

$$r^2 - r\left\{\cos(\theta - 60°) + 2\cos(\theta - 30°)\right\} + \sqrt{3} = 0.$$

Ex. 4. The centre of a circle is the point

$$(\tfrac{1}{2}\sqrt{A^2 + B^2}, \tan^{-1} B/A).$$

Find its equation.

Ex. 5. Shew that the area of the rectangle contained between the segments of the chords of a circle drawn through a given point is constant ; and is the same as the square of the tangent from the point, if the point is outside the circle.

3·11. The equation to the circle described on the line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$ (I. C. G. 7·22)

3·12. The necessary and sufficient conditions that the general equation of the second degree *viz.*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

may represent a circle are $a = b$, $h = 0$ and $g^2 + f^2 - c$ positive. (I. C. G. 7·31)

3·13. The general equation to a circle can be written in the form $x^2+y^2+2gx+2fy+c=0$, $(-g, -f)$ being its centre and $\sqrt{g^2+f^2-c}$ its radius.

3·14. A point (x', y') lies without, upon or within the circle $x^2+y^2+2gx+2fy+c=0$, according as

$$x'^2+y'^2+2gx'+2fy'+c >= \text{ or } <0. \qquad \text{(I. C. G. 7·5)}$$

3·21. The co-ordinates of any point on the circle $x^2+y^2=a^2$ in terms of a single variable θ are $(a \cos \theta, a \sin \theta)$. For brevity the point is called 'θ.'

(I. C. G. 9·50).

3·22. The equation to the chord joining two points 'θ_1' and 'θ_2' on the circle $x^2+y^2=a^2$ is

$$x \cos \frac{\theta_1+\theta_2}{2}+y \sin \frac{\theta_1+\theta_2}{2}=a \cos \frac{\theta_1-\theta_2}{2}$$

(I. C. G. 9·51).

3·31 The straight line $y=mx+c$ intersects the circle $x^2+y^2=a^2$ in real and different, coincident or imaginary points according as

$$c^2 <= \text{ or } > a^2(1+m^2) \qquad \text{(I. C. G. 8·20)}$$

3·32 If l be the length of the chord intercepted by this circle on the line $y=mx+c$, then

$$l=2\sqrt{a^2(1+m^2)-c^2}/\sqrt{1+m^2} \qquad \text{(I. C. G. 8·20)}$$

3·33. (α) The straight line $y=mx+a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$ for all values of m, the point of contact being $\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$

(I. C. G. 8·20 and 9·20.)

(β) The equation to the tangent at the point (x', y') to the circle

(i) $x^2+y^2+2gx+2fy+c=0$ is
$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

(ii) $x^2+y^2=a^2$ is $xx'+yy'=a^2$. (I. C. G. 9·1)

The equation of (α) could be deduced from (ii) by substituting $-\dfrac{x'}{y'}=m$

$$or \quad x'= -\frac{am}{\sqrt{1+m^2}},\ y'=\frac{a}{\sqrt{1+m^2}}, \text{[because } x'^2+y'^2=a^2]$$

(γ) The equation to the tangent at 'θ' follows from the equation of Art 3·22 by putting $\theta_2=\theta_1=\theta$. It is

$$x \cos\theta + y \sin\theta = a. \qquad \text{(I. C. G. 9·51)}$$

The same could also have been obtained by putting $m= -\cos\theta$ in the equation of (α) or by putting

$x=a\cos\theta$, $y=a\sin\theta$ in the equation (β) (ii)

(δ) The point of intersection of the tangents at θ_1 and θ_2 is

$$\left[a\cos\frac{\theta_1+\theta_2}{2}\Big/\cos\frac{\theta_1-\theta_2}{2},\ a\sin\frac{\theta_1+\theta_2}{2}\Big/\cos\frac{\theta_1-\theta_2}{2}\right]$$

3·34. The equation to the normal to the circle $x^2+y^2+2gx+2fy+c=o$ at any point (x', y') is

$$x\,(f+y')-y\,(g+x')-fx'+gy'=o \qquad \text{(I. C. G. 9·3)}$$

3·35. The length of the tangent from any point (x', y') to the circle $x^2+y^2+2gx+2fy+c=o$ is

$$\sqrt{x'^2+y'^2+2gx'+2fy'+c}. \qquad \text{(I. C. G. 9·40)}$$

3·41. Through any point two tangents, one tangent, or no tangents can be drawn to a circle according as the point lies outside, upon or within the circle. (I. C. G. 10·10)

3·42. The equation to the chord of contact of the tangents drawn from any point (x', y') to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$xx'+yy'+g(x+x')+f(y+y')+c=0. \quad \text{(I. C. G. 10·20)}$$

3·43. The equation to the pair of tangents drawn from any point (x', y') to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$(x^2+y^2+2gx+2fy+c)(x'^2+y'^2+2gx'+2fy'+c)$$
$$=\{xx'+yy'+g\,(x+x')+f\,(y+y')+c\}^2$$

or

$$SS'=T^2 \qquad \text{(I. C. G. 10·3)}$$

3·51. The polar of a point (x', y') with respect to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$xx'+yy'+g(x+x')+f(y+y')+c=0 \qquad \text{(I. C. G. 11·10)}$$

3·52. If the polar of a point P passes through Q then the polar of Q passes through P. Points so connected are called *conjugate points*. (I. C. G. 11·4)

3·53 If the pole of a line AB lies on CD, then the pole of CD lies on AB. Straight lines so connected are called *conjugate lines*.

3·61. *To find the locus of the middle points of a system of parallel chords $y=mx+\lambda$ (where m is constant for the system and λ is variable) of the circle $x^2+y^2+2gx+2fy+c=0$.*

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the point of intersection of the chord $y=mx+\lambda$ with the given circle, then the two abcissae are the roots of x in

$$x^2+(mx+\lambda)^2+2gx+2f\,(mx+\lambda)+c=0.$$

or $x^2(1+m^2)+2x(m\lambda+g+mf)+\lambda^2+2f+c=0.$

$$\therefore\ x_1+x_2=\quad -\frac{2(m\lambda+g+mf)}{1+m^2}$$

Also $\quad y_1+y_2= \quad m(x_1+x_2)+2\lambda= \quad \dfrac{2\,(\lambda-gm-fm^2)}{1+m^2}$

If (h, k) be the middle point of the chord

$$h=\frac{x_1+x_2}{2}= \quad -\frac{m\lambda+g+mf}{1+m^2}$$

$$\text{or} \qquad h+\frac{g+mf}{1+m^2}= \quad -m\lambda/1+m^2 \ldots\ldots\ldots\ldots\ldots(1)$$

$$k=\frac{y_1+y_3}{2}=\frac{\lambda-gm-fm^3}{1+m^2}$$

$$\text{or} \qquad k+\frac{gm+fm^2}{1+m^2}=\lambda/1+m^2 \ldots\ldots\ldots\ldots\ldots(2)$$

Eliminating λ between (1) and (2) we get

$$\left(h+\frac{g+mf}{1+m^2}\right)+m\left(k+\frac{gm+fm^2}{1+m^2}\right)=0.$$

$$\text{or} \quad (h+g)+m(k+f)=0.$$

Generalising the co-ordinates h and k, the required locus is
$$x+g+m(y+f)=0.$$

Obviously the above equation represents a straight line passing through $(-g, -f)$, the centre of the circle, and perpendicular to the given system of parallel chords. It is called a *diameter* of the conic.

Every straight line passing through the centre of a circle is thus a diameter, the straight lines at right angles to it being the system of the parallel chords which it bisects.

If the equation to the circle be in the simpler form $x^2+y^2=a^3$, the corresponding *diameter* will be the straight line
$$x+my=0.$$

3·62. *To find the equation to the chord of the circle $x^2+y^2=a^2$, having (x_1, y_1) for its middle point.*

Let the equation to the chord be

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1) \text{ (Art. 2·14)}$$

The co-ordinates of any point on this chord are

$$\left.\begin{aligned} x&=x_1+r\cos\theta \\ y&=y_1+r\sin\theta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting from (2) in the equation to the circle

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2$$

or $r^2+2r(x_1\cos\theta+y_1\sin\theta)+x_1^2+y_1^2-a^2=0$ ....(3)

If (x_1, y_1) is the middle point of the chord, the two values of r in (3) must be equal in magnitude, but opposite in sign or their algebraic sum must be zero *i.e.*

$$x_1\cos\theta+y_1\sin\theta=0$$

$$\text{or} \quad \frac{\cos\theta}{y_1}=-\frac{\sin\theta}{x_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Substituting for $\cos\theta$ and $\sin\theta$ in (1) from (4) we get

$$\frac{x-x_1}{y_1}+\frac{y-y_1}{x_1}=0$$

or $\quad xx_1+yy_1=x_1^2+y_1^2$, the required equation.

If the equation to the circle were in the form

$$x^2+y^2+2gx+2fy+c=0,$$

the equation to the chord could similarly be shown to be

$$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1.$$

Ex. I. Find the locus of the middle points of the chords of the circle $x^2+y^2+2gx+2fy+c=0$, which subtend a right angle at the origin.

Ex. 2. Show that the tangents at the extremities of a diameter are parallel to the system of chords bisected by that diameter.

3·71. In order that two circles may touch, the distance between their centres must be equal to the sum (external touch) or the difference (internal touch) of their radii. (I. C. G. 12·1.)

3·72. Two circles intersect orthogonally if the sum of the squares of their radii is equal to the square of the distance between their centres.

If these circles are

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0\ ;$$

the condition for their orthogonal intersection is

$$2g_1\,g_2+2f_1\,f_2=c_1+c_2. \qquad \text{(I. C. G. 12·2.)}$$

3·73. *To find the locus of a point which moves so that the lengths of the tangents from it to two given circles*

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $x^2+y^2+2g_2x+2f_2y+c_2=0$ *are equal.*

Let the moving point be (h, k).

By hypothesis

$$h^2+k^2+2g_1h+2f_1k+c_1=h^2+k^2+2g_2h+2f_2k+c_2$$

or $$2h(g_1-g_2)+2k(f_1-f_2)+c_1-c_2=0$$

Generalising the co-ordinates h and k, the required locus is seen to be a straight line given by

$$2x\,(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0.$$

This straight line is called the **Radical Axis** of the two circles. A glance at its 'm' will show that it is at right angles to the line joining their centres.

3·74. If

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1$$
$$\text{and } S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2\,;$$

the radical axis of the circles given by $S_1=0$ and $S_2=0$ is seen (Art. 3·73) to be S_1-S_2*$=0$.

Thus the radical axis passes through the points common to the two circles. If the circles cut, it is their common chord. If the circles touch, it is their common tangent. If they neither cut nor touch, it may be said to pass through their imaginary points of intersection.

The above is a very important property of the radical axis, so important, in fact, that it may be taken as the definition itself. Thus the *Radical Axis* of two circles may be defined as *the straight line which passes through their common points.*

Ex. 1. Find the equations to the radical axes of the circles :—

(i) $x^2+y^2+4y-4x-1=0$
and $x^2+y^2+6x-3y-1=0$

(ii) $2x^2+2y^2+14x-18y+15=0$
and $4x^2+4y^2-3x-y+5=0$

Ex. 2. Find the equations to the common chord of the two circles $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$.

Also find the length of this common chord, and show that the circle described on it as diameter is

$$(a^2+b^2)(x^2+y^2)=2ab\,(bx+ay).$$

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*It should be carefully noted that if the co-efficient of x^2 and y^2 in S_1 and S_2 are not the same *i.e.* if $S_1 \equiv Ax^2+Ay^2+2G_1x+2F_1y+C_1$ and $S_2 \equiv Bx^2+By^2+2G_2x+2F_2y+C_2$, the radical axis of $S_1=0$ and $S_2=0$ will not be given by $S_1-S_2=0$, but by

$$\frac{S_1}{A}-\frac{S_2}{B}=0 \quad \text{or} \quad BS_1-AS_2=0.$$

Ex. 3. Shew that the locus of the points, such that the difference of the squares on tangents from them to two given circles is constant, is a straight line parallel to the radical axis of these circles.

3·75. *To prove that the radical axes of three circles taken in pairs meet in a point.*

Let the three circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0$$

$$\text{and } S_3 \equiv x^2+y^2+2g_3x+2f_3y+c_3=0$$

The three radical axes are

$$S_1-S_2=0,\ S_2-S_3=0 \text{ and } S_3-S_1=0$$

Adding up these three equations the result is identically zero.

Hence the three straight lines meet in a point.

(Art. 2·53)

Such a point is called the **Radical Centre** of the three circles, and is such that the lengths of the tangents from it to the three circles are equal.

Ex. 1. Find the radical centre of the set of circles.

(i) $3x^2+3y^2-4x-6y-1=0$, $2x^2+2y^2-3x-2y-4=0$ and $2x^2+2y^2-x+y-1=0$

(ii) $x^2+y^2-3x-6y+8=0$, $x^2+y^2-x-4y+2=0$ and $x^2+y^2+2x+6y+3=0$.

Ex. 2. Find the point from which the tangents to the three circles

$$x^2+y^2-16x+60=0,\ x^2+y^2-12x+27=0 \text{ and } x^2+y^2-12y+84=0.$$

are equal in length, and find that length.

3·81. In Art. 3·74 will be found an interpretation of the equation $S_1+\lambda S_2=0$ for $\lambda=-1$. Let us now interpret it for $\lambda \neq -1$.

Co-ordinates that satisfy $S_1=0$ and $S_2=0$ simultaneously, satisfy $S_1+\lambda S_2=0$. Also the latter is an equation of the second degree in x and y, in which the co-efficients of x^2 and y^2 are $(1+\lambda)$ each, and the co-efficient of xy is zero. Hence it represents a circle passing through the common points, real and different, coincident or imaginary of $S_1=0$ and $S_2=0$. Thus for varying values of $\lambda(\neq -1)$, the given equation represents a system of circles all passing through the same two points, and therefore having the same radical axis.

Let $S_1+\lambda_1 S_2=0$ and $S_1+\lambda_2 S_2=0$ be any two circles of this system. Their radical axis is at once seen to be $S_1-S_2=0$.* The nature of this result being independent of λ shows that the radical axis of any pair of circles of the system is the same. Such a system is known as a **Coaxal System.**

The circles $S_1=0$ and $S_2=0$ belong to the system.

Obviously if $S=0$ and $L=0$ represent respectively a circle and the radical axis of a Coaxal System, the system itself is given by $S+\lambda L=0$.

Ex. 1. Find the equation of the system of circles having the same radical axis as

$$x^2+y^2=25$$
$$\text{and } (x-1)^2+(y-1)^2=27.$$

Ex. 2. Find the equations of the circles touching the straight line $x+y=5$ and coaxal with the circles

$$x^2+y^2-6x-6y+4=0$$
$$\text{and } x^2+y^2-2x-4y+3=0.$$

*See footnote to Art. 3·74.

Ex. 3. Find the equation of the circle coaxal with

$$x^2+y^2+2gx+2fy+c=0 \text{ and } x^2+y^2+2g'x+2f'y+c'=0$$

and passing through the origin.

Ex. 4. Find the equation of the circle which passes through the points of intersection of $x^2+y^2+4x-8y+4=0$ and $x^2+y^2+2x+2y-2=0$, and also through the origin.

Ex. 5. Find the equation of the circles which have the line $x-y=0$ for their radical axis.

Ex. 6. Find the equation of a system of circles having the straight line $3x-5y=7$ for their radical axis, one circle of the system having the origin for its centre and 4 for its radius.

3·82. The centres of the circles $S_1+\lambda_1 S_2=0$ and $S_1+\lambda_2 S_2=0$ of the last article are

$$\left(-\frac{g_1+\lambda_1 g_2}{1+\lambda_1}, -\frac{f_1+\lambda_1 f_2}{1+\lambda_1}\right) \text{ and } \left(-\frac{g_1+\lambda_2 g_2}{1+\lambda_2}, -\frac{f_1+\lambda_2 f_2}{1+\lambda_2}\right)$$

respectively.

The equation to the straight line joining them will be seen to be

$$x(f_1-f_2)-y(g_1-g_2)=g_1 f_2-g_2 f_1$$

The 'm' of this straight line is $\dfrac{f_1-f_2}{g_1-g_2}$. Also its equation is independent of λ. Hence the centres of all the circles of a Coaxal System lie on a straight line at right angles to the radical axis.

3·83. *To find in its simplest form the equation to a system of Coaxal Circles.*

If the line of centres of these circles be taken as the axis of x, the equation to any circle of the system will be of the form

$$x^2+y^2+2gx+c=0 \quad \dots\dots\dots(1)$$

where g and c are arbitrary constants.

Let two circles of the system be

$$x^2+y^2+2g_1x+c_1=0$$
$$\text{and } x^2+y^2+2g_2x+c_2=0.$$

Their radical axis is

$$2x(g_1-g_2)+c_1-c_2=0 \quad \text{......................(2)}$$

Being perpendicular to the line of centres (Art. 3·83) it can be taken as the axis of y, whence c_1-c_2 must be zero *i.e.* c is the same for all circles of the system and ceases to be arbitrary.

Thus $x^2+y^2+2gx+c=0$, where only g is a variable parameter, is the simplest form of the equation to a Coaxal System.

The equation is often written in the form

$$x^2+y^2-2\lambda x+c=0,$$

where λ is a variable parameter, and c a constant.

Ex. 1. Reduce the equations $x^2+y^2+2x+6y+9=0$ and $x^2+y^2-6x+8=0$ to their simplest forms.

Ex. 2. Shew that the locus of a point which moves so that the tangents from it to two given circles are in a constant ratio, is a coaxal circle.

Ex. 3. From any point distant c from the origin, tangents are drawn to three of the coaxal circles $x^2+y^2-2\lambda x-c^2=0$. Shew that the lengths of the tangents are in Geometrical Progression, if the distances of their centres from the origin are so.

Ex. 4. Prove that in any coaxal system, there are two circles real, coincident or imaginary which touch a given straight line.

3·91. The equation to the coaxal system given by $S_1+\lambda S_2=0$ can be put down as

$$\left(x+\frac{g_1+\lambda g_2}{1+\lambda}\right)^2+\left(y+\frac{f_1+\lambda f_2}{1+\lambda}\right)^2$$
$$=\left(\frac{g_1+\lambda g_2}{1+\lambda}\right)^2+\left(\frac{f_1+\lambda f_2}{1+\lambda}\right)^2-\frac{c_1+\lambda c_2}{1+\lambda} \quad \text{.........(1)}$$

The λ for point-circles of the system *i.e.*, the circles with radius zero is, therefore, given by

$$(g_1+\lambda g_2)^2+(f_1+\lambda f_2)^2-(c_1+\lambda c_2)(1+\lambda)=0$$

or $$\lambda^2(g_2^2+f_2^2-c_2)+\lambda(2g_1g_2+2f_1f_2-c_1-c_2)$$
$$+g_1^2+f_1^2-c_1=0. \quad \ldots\ldots\ldots(2)$$

These point circles are called the **'Limiting points'** of the system and are from (1) and (2) seen to be

$$\left(-\frac{g_1+\lambda_1g_2}{1+\lambda_1}, -\frac{f_1+\lambda_1f_2}{1+\lambda_1}\right) \text{ and } \left(-\frac{g_1+\lambda_2g_2}{1+\lambda_2}, -\frac{f_1+\lambda_2f_2}{1+\lambda_2}\right)$$

where λ_1 and λ_2 are the roots of λ in equation (2).

To find, therefore, the limiting points of a system of coaxal circles

(*i*) *Put the equation to the system in the form*

$$(x-h)^2+(y-k)^2=r^2.$$

(*ii*) *Find the values of the parameter for which* $r=0$.

(*iii*) *Evaluate* h *and* k *for each of these values.*

Each point (h, k) *is a limiting point.*

Ex. 1. **Find the limiting points of the coaxal system of which the circles $x^2+y^2+2x+4y+7=0$ and $x^2+y^2+4x+2y+5=0$ are the two members.**

The equation of the corresponding coaxal system is

$$x^2+y^2+2x+4y+7+\lambda\,(x^2+y^2+4x+2y+5)=0$$

or $$x^2+y^2+2x\left(\frac{1+2\lambda}{1+\lambda}\right)+2y\left(\frac{2+\lambda}{1+\lambda}\right)+\frac{7+5\lambda}{1+\lambda}=0.$$

or $$\left(x+\frac{1+2\lambda}{1+\lambda}\right)^2+\left(y+\frac{2+\lambda}{1+\lambda}\right)^2=\left(\frac{1+2\lambda}{1+\lambda}\right)^2$$
$$+\left(\frac{2+\lambda}{1+\lambda}\right)^2-\left(\frac{7+5\lambda}{1+\lambda}\right)$$

The radius of this circle is zero, if

$$(1+2\lambda)^2+(2+\lambda)^2-(7+5\lambda)(1+\lambda)=0$$

or $$(4\lambda^2+4\lambda+1)+(4+4\lambda+\lambda^2)-(7+12\lambda+5\lambda^2)=0.$$

This shows that $\lambda=\infty$ or $-\frac{1}{2}$.

The limiting points are given by

$$\left(-\frac{1+2\lambda}{1+\lambda}, \quad -\frac{2+\lambda}{1+\lambda}\right)$$

$$i.e. \quad \left\{\left(-1-\frac{\lambda}{1+\lambda}\right), \left(-1-\frac{1}{1+\lambda}\right)\right\}$$

i.e. $(-2, -1)$ and $(0, -3)$.

Ex. 2. Find the limiting points of the coaxal systems defined by the circles :—

(*i*) $x^2+y^2-6x-6y+4=0$ and $x^2+y^2-2x-4y+3=0$

(*ii*) $x^2+y^2-3x-1=0$ and $2x^2+2y^2-7x+2=0$.

3·92. *To shew that the limiting points of a system of coaxal circles are real and different, coincident or imaginary according as the system is of the non-intersecting, touching or intersecting species.*

Let the equation to the coaxal system be

$$x^2+y^2+2\lambda x+c=0.$$

where λ is the variable parameter

$$or \quad (x+\lambda)^2+y^2=\lambda^2-c.$$

For point circles of the system $\lambda= \pm\sqrt{c}$

Hence its limiting points are $(\pm\sqrt{c}, 0)$.

Also the common points of the system being the same in which any circle of the system meets the radical axis $x=0$, their ordinates are given by $y^2+c=0$.

Hence these points are $(0, \pm\sqrt{-c})$.

Evidently if c is positive the limiting points are real and different, whereas the common points are imaginary; and if c is negative, the nature of these points is reversed.

If $c=0$, the limiting points and the common points all coincide.

Ex. 1. The limiting points of a coaxal system are conjugate with respect to any circle of the system.

Ex. 2. Prove that the polar of a limiting point of the coaxal system is the same for all circles of the system, and passes through the other limiting point.

Ex. 3. Find the points of intersection and the limiting points of the coaxal systems defined by :—

(*i*) $x^2+y^2-6x-12y+55=0$ and $3x^2+3y^2-20x-42y+207=0$

(*ii*) $x^2+y^2-2x-4y-3=0$ and $x^2+y^2-12x-14y+67=0$

(*iii*) $x^2+y^2+26x-6y+153=0$ and $x^2+y^2+10x-6y+9=0$

3·93. *To find the equation to a coaxal system of circles, its limiting points being given.*

Let these limiting points be (h_1, k_1) and (h_2, k_2).

The equations to the corresponding point-circles are

$$(x-h_1)^2+(y-k_1)^2=0 \text{ and } (x-h_2)^2+(y-k_2)^2=0.$$

Since both of these circles belong to the required coaxal system, the equation to this system by art. 3·81 is

$$(x-h_1)^2+(y-k_1)^2+\lambda\{(x-h_2)^2+(y-k_2)^2\}=0.$$

Ex. 1. Find the equation to the system of coaxal circles whose limiting points are

(*i*) $(a, 0)$ and $(-a, 0)$

(*ii*) $(2, 3)$ and $(4, 9)$

(*iii*) $(-1, 2)$ and $(3, -5)$.

3·94. *To find the equation to a system of circles which intersects a given coaxal system orthogonally.*

Let the coaxal system be $S_1+\lambda S_2=0$, where S_1 and S_2 have the same significance as in Art. 3·81.

If any circle

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots(1)$$

is to cut all circles of the coaxal system orthogonally, then

$$2g\left(\frac{g_1+\lambda g_2}{1+\lambda}\right)+2f\left(\frac{f_1+\lambda f_2}{1+\lambda}\right)=c+\frac{c_1+\lambda c_2}{1+\lambda} \quad \text{(Art. 3·72.)}$$

$$\text{or} \quad \lambda(2gg_2+2ff_2-c-c_2)+2gg_1+2ff_1-c-c_1=0 \quad \text{.........(2)}$$

Equation (2) will hold good for all values of λ, only when

$$2gg_2+2ff_2-c-c_2=0 \quad \text{..................(3)}$$
$$\text{and} \quad 2gg_1+2ff_1-c-c_1=0 \quad \text{..................(4)}$$

With the help of (3) and (4) two of the three constants g, f and c in (1) can be eliminated. The resulting equation containing only one variable parameter represents the required system.

Ex. 1. Find the equation of the family of circles cutting the two circles

$$x^2+y^2+2x+4y+7=0$$
$$\text{and} \quad x^2+y^2+4x+2y+5=0$$

orthogonally, and show that they all pass through the limiting points of the coaxal system of which the two given circles are the member.

Ex. 2. Find the equation of a circle which is cut orthogonally by every one of the three circles

$$x^2+y^2-2x-4y+6=0,$$
$$x^2+y^2+4y-6=0$$
$$\text{and} \quad x^2+y^2-10x+18=0\ ;$$

and shew that its centre and radius are respectively the radical centre and the length of the tangent from the radical centre to any of the given circles.

Ex. 3. Shew that a circle having its centre at the radical centre of any three circles and its radius equal to the length of the tangent from the radical centre to any of the given circles, will cut them all orthogonally.

3·95. To facilitate the investigation of any further properties that this new system may possess we shall proceed with the simplest form of the equation to a coaxal system, *viz.*,

$$x^2+y^2-2\lambda x+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where λ is the variable parameter.

If any circle

$$x^2+y^2+2Gx+2Fy+C=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

is to cut (1) orthogonally, then

$$-2G\lambda=c+C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since (3) is to be true for all values of λ

$$G=0 \quad \text{and} \quad C=-c.$$

Substituting these values of G and C in (1), we get

$$x^2+y^2+2Fy-c=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where F is a variable parameter.

Hence (4) represents a system of circles cutting orthogonally the system represented by (1).

Since the radical axis of any two circles of (4) is the same *viz.*, $y=0$; this also is a coaxal system. Thus corresponding to any coaxal system, an orthogonal coaxal system can be found. It can be easily verified that the two systems are so related that the radical axis of one is the line of centres of the other, and the limiting points of one system are the common points of the other.

Two such systems are called **Conjugate Systems.**

Ex. 1. All circles of a coaxal system are cut orthogonally by every circle passing through the limiting points.

Ex. 2. If a circle cuts two circles of a Coaxal System orthogonally, it will cut them all orthogonally.

Examples.

1. Shew that the polars of any fixed point P with respect to a system of coaxal circles pass through another fixed point Q, and PQ subtends a right angle at each of the limiting points.

2. Find the equations of the circles on the three diagonals of the quadrilateral, the equations of whose sides are respectively $y-1=0$, $x-y+1=0$, $x+5y-11=0$ and $3x+y-13=0$, and shew that they are coaxal.

3. If the origin be at one of the limiting points of a system of coaxal circles of which

$$x^2+y^2+2gx+2fy+c=0$$

is a member, prove that the equation of the system is

$$\lambda(x^2+y^2)+2gx+2fy+c=0$$

and that the equation of the conjugate system is

$$(x^2+y^2)(g+\mu f)+c(x+\mu y)=0.$$

4. The polars of a point P with respect to two circles meet in Q. Shew that the radical axis of the circles bisects PQ.

5. PT_1 and PT_2 are tangents drawn from a given point P to two given circles of a coaxal system. Prove that as P moves round the circumference of a third circle of the same system, the ratio $PT_1 : PT_2$ is constant.

6. Prove that a common tangent to two circles of a coaxal system subtends a right angle at either limiting point of the system.

7. Prove that the locus of a point at which two given circles subtend equal angles is a coaxal circle.

8. Prove that the limiting points of the system $x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$ subtend a right angle at the origin, if $\frac{c}{g^2}+\frac{k}{f^2}=1$.

9. Prove that the equation to two given circles can always be put in the form $x^2+y^2+ax+b=0$ and $x^2+y^2+a'x+b=0$.

If A, B, C be the centres of three coaxal circles, and t_1, t_2, t_3 the lengths of the tangents to them from any point, prove that

$$BC.t_1^2+CA.t_2^2+AB.t_3^2=0. \qquad \text{[Agra. U. 1928]}$$

10. A chord PQ of a circle belonging to a coaxal system touches another circle of the same system at R. If L be a limiting point of the system, prove that

$$PR : PL :: QR : QL. \qquad \text{[Agra. U. 1933]}$$

Miscellaneous Exercises.

1. Find the equations of the circles passing through the points $(-1, 5)$ and $(8, 8)$ and touching the axis of x.

2. Find the equations of the two circles having their centres at the origin and touching the circle passing through the points $(6, 5)$ $(-1, -2)$, $(-2, 5)$.

3. The equations of two circles are $x^2+y^2=4$ and $x^2+y^2-10x-14y+65=0$. Find the equation of a straight line, inclined to the axis of x at $45°$ and such that the circles intercept equal chords on it.

4. $(2, 7)$ and $(5, 3)$ are the extremities of the diameter of a circle. Find the equation to the tangents of this circle which are parallel to the line $5x-12y+2=0$.

Shew that one of them also touches the circle $x^2+y^2=6y$ and give the equation of the circle concentric with the latter to which the other one is a tangent.

5. Shew that the circle described on the straight line joining $(am^2, 2am)$ and $(a/m^2, -2a/m)$ as diameter touches the line $x+a=0$.

6. Prove that the normal to the circle

$$x^2+y^2-5x+2y-48=0$$

at the point (5, 6) touches the circle

$$x^2+y^2+28x+6y-16=0.$$

Find the co-ordinates of the point of contact and its distance from (5, 6)

7. Find the equation of the circle inscribed in the triangle formed by the lines $3x+4y=12$, $3x-4y=36$ and $x=0$. [*Alld. U. 1937*]

8. OQ is drawn from the fixed point O to meet a fixed straight line AQ at Q. In OQ a point P is taken, so that $OP.OQ=a^2$. Find the locus of P, using polar co-ordinates.

9. O is a fixed point on a circle OPB of radius a, and OP is produced to Q, so that $OQ=k.OP$. Find the locus of Q.

10. If $x/a+y/b=1$ be the equation to a chord of the circle $x^2+y^2=c^2$, shew that the pair of tangents at its extremities intersects in a point $(c^2/a, c^2/b)$ and also that if t_1 and t_2 are the tangents of the angles which these tangents make with the axis of x, then

$$\frac{1}{t_1}+\frac{1}{t_2}=\frac{2bc^2}{a(c^2-b^2)}.$$

11. Shew that the locus of a point which moves in such a manner that the length of the tangent from it to

the circle $x^2+y^2+2gx+2fy+c=0$ is twice its distance from the origin, is a circle.

12. Find the locus of the feet of perpendiculars from the origin to tangents to the circle $x^2+y^2+4x-2y=4$.

13. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$. Prove that the area of the triangle formed by them and their chord of contact is

$$\frac{a\,(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$$

14. Find the common tangents to the circles $x^2+y^2+4x+2y-4=0$ and $x^2+y^2-4x-2y+4=0$.

15. Prove that the locus of the poles of the tangents to the circle

$$(x-b)^2+y^2=c^2$$

with respect to the circle

$$x^2+y^2=a^2$$

is $$(c^2-b^2)x^2+c^2y^2+2\,a^2bx-a^4=0.$$

16. Circles are drawn through the point $(c, 0)$ touching the circle $x^2+y^2=a^2$. Shew that the locus of the pole of the axis of x with respect to these circles is the curve

$$4a^2(x-c)^4=(a^2-c^2)\{a^2-(c-2x)^2\}y^2.$$

17. If the chord of contact of tangents drawn from an external point to the circle whose equation is

$$(x-a)^2+(y-b)^2=c^2$$

be perpendicular to the diameter through the origin, shew that the point lies on the straight line whose equation is

$$ax-by=a^2-b^2.$$

18. Shew that the two circles $x^2+y^2-4x-6y+11=0$ and $x^2+y^2-10x-4y+21=0$ cut one another orthogonally.

Also find the equation of a circle which cuts the given circles at right angles and has $2x+3y=7$ as a diameter.

19. If the pairs of opposite sides of a quadrilateral be formed by the straight lines whose equation are

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

prove that $\dfrac{h}{h'}=\dfrac{a-b}{a'-b'}$.

20. Shew that the difference of the squares of the tangents from any point to two circles is proportional to the distance of this point from their radical axis.

21. Shew that the square of the tangent which can be drawn from a point on one circle to another is proportional to the distance of the point from their radical axis.

22. Find the length of the common chord of the circle $(x-a)^2+(y-b)^2=c^2$ and $(x-b)^2+(y-a)^2=c^2$, and hence prove that if these circles touch each other, then $$2c^2=(a-b)^2.$$

23. If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third.

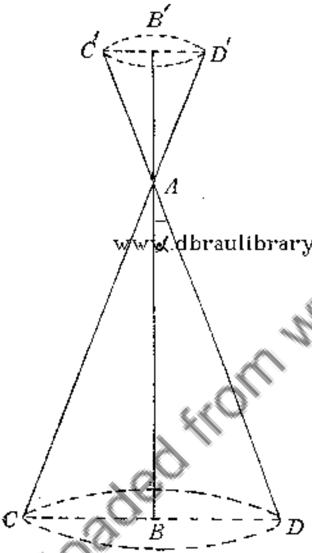
24. Find the equation of the circles of radius 7 which pass through the common points of the circle $x^2+y^2=24$ and the straight line $3x-4y=0$.

25. Find the equation of the circle passing through the origin and the intersections of $x^2+y^2+bx-8y+4=0$ and $3x-2y+2=0$.

CHAPTER IV.

CONICS.

4·10. A curve in which a double right circular cone is intersected by a plane is called a conic. For different positions of the intersecting plane these curves will be different. Since, however, they are derived from the cone they are all classed as conics.



Let $ACDC'D'$ be such a cone whose axis is BAB' and whose semi-vertical angle is α. Let it be intersected by a plane making an angle θ with the axis. Obviously there are two cases according as the intersecting plane does not or does pass through the vertex of the cone.

Case I. *The plane does not pass through the vertex of the cone.*

The section is

(*i*) For $\theta=\alpha$, a curve called parabola.

(*ii*) For $\theta>\alpha$, a curve called ellipse which reduces to a circle if $\theta=90°$.

and (*iii*) For $\theta<\alpha$, a curve called hyperbola, which has obviously two distinct branches one on each cone.

Case II. *The plane passes through the vertex.* The section is a pair of straight lines, real and different for $\theta < \alpha$ and coincident for $\theta = \alpha$. For $\theta > \alpha$ it may be looked upon as a pair of imaginary straight lines intersecting in real point, *i.e.* the vertex.

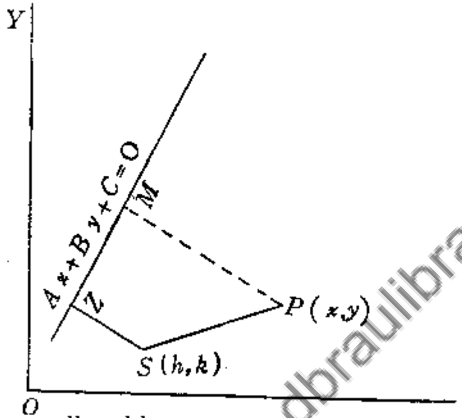
Thus in whatever way a cone may be intersected by a plane, the section is one of the five curves *viz.* a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. These, therefore, are the five and only five conics. Since, however the straight lines of Case II may be looked upon only as limiting forms of the curves of Case I, and the circle has been shown to be only a special case of the ellipse, the main conics are only three in number *viz.* the parabolas, the ellipses, and the hyperbolas.

4·11. The definition of the conics given in the last Art. is evidently of no use to us in a geometry of two dimensions. It stands to reason, however, that the strong family tie between these curves (since they have the same parentage) must be reflected in some analytical property which may serve as a definition for our purposes. In fact it is seen* that all of them possess the focus-directrix property i.e. *are the loci of points which move in such a manner that their distances from a fixed point, called the focus, always bear a constant ratio to their distances from a fixed straight line called the directrix.* This constant ratio is designated as *eccentricity* and is generally denoted by e.

Let $Ax+By+C=0$ be the equation to the directrix, (h, k) the co-ordinates of the focus S_1 and (x, y) the

*The proof is beyond the scope of this book.

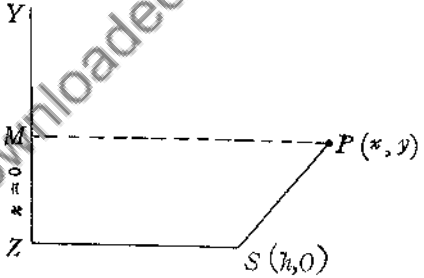
moving point P. Let the feet of the perpendiculars from



these points on the directrix be Z and M respectively. The equation to the corresponding conic is obviously

$$(x-h)^2+(y-k)^2=e^2\frac{(Ax+By+C)^2}{A^2+B^2} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

since $SP^2=e^2PM^2$.



Taking the directrix as the axis of y, and the straight line SZ as the axis of x, the above equation reduces to

$$(x-h)^2+y^2=e^2x^2. (2)$$

which is a simple form of the equation to a conic.

Now two cases arise according as the focus is not or is situated on the directrix.

Case I. *The focus is not situated on the dirextrix i.e.* $h \neq 0$.

The locus is

(i) For $e=1$ a curve called parabola.

(ii) For $e<1$ a curve called ellipse which reduces to a circle if $e=0$*.

and (iii) For $e>1$ a curve called a hyperbola.

Case II. *The focus is situated on the directrix i.e.* $h=0$.

The locus is a pair of straight lines, real and different for $e<1$, coincident for $e=1$ and imaginery for $e \gtrless 1$.

Thus it is seen that the focus-directrix property is a characteristic of five curves and five only. *viz.* a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. Also since, as in the last Art. the sraight lines of Case II are only limiting forms of the curves of Case I and the circle is only a special case of the ellipse, the parabola, ellipses and hyparbolas are the three main curves possessing this property.

4·20. *To find the 'simplest' or the standard form of the equation to a parabola.*

Putting $e=1$ in equation (2) of the last Art., we get,

$$y^2 = 2h\left(x - \frac{h}{2}\right)$$

Transferring the origin to the point $\left(\frac{h}{2}, o\right)$ *i.e.* to a point A midway between S and Z, it further reduces to

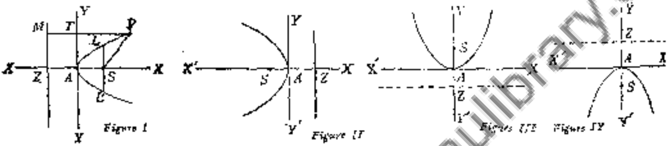
$$y^2 = 2hx.$$

*This is apparent from equation (2) of Art. 4·11.

The distance between the focus and the directrix in the case of a porabola is generally denoted by $2a$, *i.e.* $h=2a$. Hence the required equation is $y^2=4ax$.........(1)

The point A is called the vertex of the parabola and ZS is called the axis.

The shape of this parabola is given in Fig. I.



For shapes as in fig. II, III and IV, the corresponding equations are

$$y^2=-4ax.$$
$$x^2=4ay.$$
$$\text{and}\quad x^2=-4ay.$$

It should be carefully noted that the Standard form of the equation to the parabola implies that the axis and the tangent at the vertex have been taken as the axes of co-ordinates.

4·21. *To find the simplest or the standard form of the equation to an ellipse* $(e<1)$.

Equation (2) of Art. 4·11 may be written as.

$$\left(x-\frac{h}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\frac{h^2e^2}{(1-e^2)^2}. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Transferring the origin to a point $C\left(\frac{h}{1-e^2},\ o\right)$ the above equation takes a still simpler form *viz.*

$$\frac{x^2}{\frac{e^2h^2}{(1-e^2)^2}}+\frac{y^2}{\frac{h^2e^2}{1-e^2}}=1. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

A question naturally arises as to where this point C is situated *i. e.*, what the geometrical significance of $\frac{h}{1-e^2}$ is ! Let A and A' be the points which divide the line ZS internally and externally in the ratio of $1 : e$. Their co-ordinates (referred to Z as the origin) then are $\left(\frac{h}{1+e}, o\right)$ and $\left(\frac{h}{1-e}, o\right)$. C is evidently the point midway between the points A and A'.

The distance AA' in the case of an ellipse is generally denoted by $2a$,

$$\therefore \frac{h}{1-e}-\frac{h}{1+e}=2a, \quad \text{or} \quad \frac{eh}{1-e^2}=a.$$

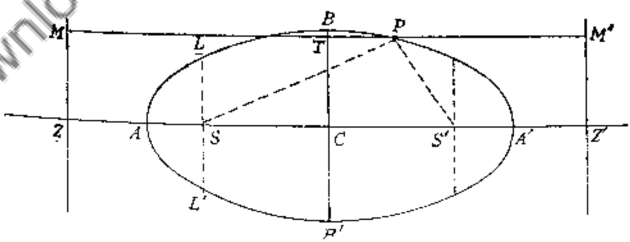
Hence the required equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $b^2=a^2(1-e^2)$.

The shape of this ellipse is



(*i*) By symmetry it is seen to have a second focus and a second directrix.

(*ii*) It may also be observed that

$$CZ=\frac{h}{1-e^2}=\frac{a}{e}$$

$$\text{and } CS=\frac{h}{1-e^2}-h=\frac{he^2}{1-e^2}=ae.$$

(*iii*) The point C is called the centre* of the ellipse. $AA'=2a$ and $BB'=2b$ are called the major and minor axes respectively.

(*iv*) *It should be carefully noted that the foci of an ellipse lie along its major axis.*

4·22. *To find the simplest or the standard form of the equation to a hyperbola* ($e>1$).

Since e is greater than unity in this case the equation (3) of the last Art., may be put down as

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1 \qquad \text{......(1)}$$

$$\textit{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad \text{......(2)}$$

$$\text{where } b^2=a^2(e^2-1).$$

which is the required equation to the hyperbola.

4·23. To trace the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

The above equation can be written both as

$$x=\pm a\sqrt{1+\frac{y^2}{b^2}}$$

$$\text{and} \qquad y=\pm b\sqrt{\frac{x^2}{a^2}-1}$$

*By the centre of the curve is understood a point such that all chords of the curve passing through that point are bisected there.

The following conclusions are apparent :—

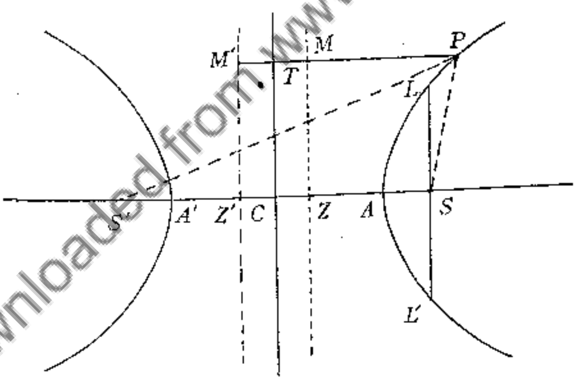
(*i*) x can not be numerically less than a. Also for $x = \pm a$, $y=0$.

Therefore the curve does not lie between $x=a$ and $x=-a$.

(*ii*) Straight lines $x=\pm a$ are tangents at $(\pm a, 0)$

(*iii*) As x increases from a (numerically), y increases, and for every value of x there are two values of y equal in magnitude but opposite in sign. Similar is the case for x as y increases from b. Thus there is symmetry about both axes.

Hence the curve is as shown below :—



Like the ellipse, for hyperbola too

(*i*) there exists a second focus and a second directrix.

(*ii*) $CZ = \dfrac{a}{e}$ and $CS = ae$.

(*iii*) The point C is called the centre* of the hyperbola. $AA'=2a$ is called its transverse axis and $BB'=2b$ its conjugate axis. The terms major and minor axes are not used in this case, because unlike the ellipse AA' is not necessarily greater than BB'.

It should be carefully noted that the hyperbola does not cut its conjugate axis, also that the ellipse lies wholly within the rectangle formed by $x=\pm a$ and $y=\pm b$ whereas the hyperbola lies wholly without it.

(*iv*) The foci always lie along the transverse axis *i.e.* the axis which cuts the hyperbola.

A hyperbola is said to be equilateral when its transverse and conjugate axes are equal *i.e.* when $b=a$. It is also called a rectangular† hyperbola. The eccentricity of such a hyperbola is evidently $\sqrt{2}$.

4·24. The portion LL' of a straight line through the focus S and perpendicular to the axis SZ intercepted between the conic is called its *Latus rectum*. The semi-latus rectum in each case is evidently the ordinate of the point on the conic where the abscissa is the same as that of the focus. Thus it is

(*i*) $2a$ for the parabola $y^2=4ax$. [Focus $(a, 0)$]

(*ii*) $\dfrac{b^2}{a}$ for the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

where $b^2=a^2(1-e^2)$. [Focii $(\pm ae, 0)$]

*See footnote page 56.

†The significance of this name will be explained afterwards.

(*iii*) $\frac{b^2}{a}$ for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

where $b^2=a^2(e^2-1)$. [Focii $(\pm ae, 0)$]

If P be any point on a conic, SP is called its *focal distance.*

(*i*) For the parabola $y^2=4ax$.

$SP=PM=PT+TM=a+x$.

(See Fig. 1. Art. 4·20.)

(*ii*) For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$SP=e.\,PM=e(PT+TM)=a+ex$.

$S'P=e.\,PM'=e(TM'-PT)=a-ex$.

(See Fig. Art. 4·21.)

Thus $SP+S'P=2a$ i.e. the sum of the focal distances of any point on an ellipse is constant and equal to the major axis.

This is a very important property of the ellipse and is sometimes taken as its definition. It gives a very convenient method for constructing the ellipse mechanically.

(*iii*) For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$SP=e\cdot PM=e(PT-TM)=ex-a$

$S'P=e\cdot PM'=e(PT+TM')=ex+a$.

(See Fig. Art. 4·23.)

Thus $S'P-SP=2a$, *i.e. the difference between the focal distances of any point on a hyperbola is constant and equal to the transverse axis.*

This again is a very important property of the hyperbola and is sometimes used as its definition.

4·25. We have seen that the equation of a conic, no matter what the nature of this conic may be *i.e.* whether it is a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola, is invariably of the second degree in x and y. We shall now proceed to demonstrate the converse proposition *viz.* that an equation of the second degree in x and y invariably represents a conic, the nature of the curve depending of course upon the constants involved in the equation. But before doing so, we shall establish a Lemma which is required for the above demonstration.

Lemma :—If the axes of co-ordinates are turned through any angle θ, origin remaining the same, the expression $ax^2+2hxy+by^2$ is transformed to $Ax^2+2Hxy+By^2$, where

$$a+b=A+B$$
$$\text{and} \quad ab-h^2=AB-H^2$$

Also if $\tan 2\theta=\dfrac{2h}{a-b}$, the transformed expression has no term containing xy i.e. $H=0$.

Now x and y are to be replaced by $(x\cos\theta-y\sin\theta)$ and $(x\sin\theta+y\cos\theta)$ respectively.

Hence $ax^2+2hxy+by^2$ is transformed into

$$a(x\cos\theta-y\sin\theta)^2+2h(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$$
$$+b(x\sin\theta+y\cos\theta)^2.$$

If this be written as $Ax^2+2Hxy+By^2$,

then

$$2A=(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta$$
$$2H=(b-a)\sin 2\theta+2h\cos 2\theta$$
$$2B=(a+b)-(a-b)\cos 2\theta-2h\sin 2\theta.$$

Evidently $A+B=a+b.$

$$4(AB-H^2)=(a+b)^2-\{(a-b)\cos 2\theta+2h\sin 2\theta\}^2$$
$$-\{(b-a)\sin 2\theta+2h\cos 2\theta\}^2$$
$$=4(ab-h^2)$$

Further obviously for $tan\ 2\theta = \dfrac{2h}{a-b}$, H is seen to be zero.

The expressions $a+b$ and $ab-h^2$ are known as **Invariants,** since they are not changed by turning the axes.

Ex. 1. If (x, y) and (x', y') be the co-ordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux+vy$ where u and v are independent of x and y becomes $u'\,x'+v'\,y'$, shew that $u^2+v^2=u'^2+v'^2$.

Ex. 2. If (x, y) and (x', y') have the same meaning as in the previous example, and if $x=kx'+ly'$ and $y=k'\,x'+l'\,y'$ shew that

$$kl'=k'l.$$

Ex. 3. Establish the property of Invariants by turning the axes through any angle θ in the following cases :—

(*i*) $6x^2-4xy+9y^2-24x-22y+43=0$

(*ii*) $4x^2+12xy+9y^2-2x+10y+21=0$

(*iii*) $4x^2+12xy-y^2-40x-20y+24=0$.

Determine the value of θ in each case such that the term containing xy may vanish.

4·26. *To shew that the general equation of the second degree in x and y, viz. $ax^2+2hxy+by^2+2gx+2fy+c=0$ always represents a conic.*

It has been shewn in Art. 2·84 above that the equation represents a pair of straight lines, if the constants involved are related such that

$$\triangle \equiv abc+2fgh-af^2-bg^2-ch^2=0.$$

Thus the proposition has to be proved further when $\triangle \neq 0$.

Let the axes be turned through an angle θ such that $\tan 2\theta = \frac{2h}{a-b}$. Thus the transformed equation will have no term containing xy.

Let the given equation take the form

$$Ax^2+By^2+2\,Gx+2\,Fy+C=0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Such that $A+B=a+b \ldots\ldots\ldots\ldots(\alpha)$

and $AB=ab-h^2 \ldots\ldots\ldots\ldots(\beta)$

Equation (1) can also be put down as

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{G^2}{A}+\frac{F^2}{B}-C=K^*\text{ (say)} \quad (2)$$

Now three cases arise according as one of the quantities A or B is zero, both of them are of the same sign or of opposite signs.

Case I. *One of the quantities A or B, say B is zero.*

Equation (1) takes the form.

$$Ax^2+2\,Gx+2\,Fy+C=0.$$

$$or \quad A\left(x+\frac{G}{A}\right)^2=-2\,Fy+\frac{G^2}{A}-C.$$

$$=-2F\left(y-\frac{G^2-AC}{2F}\right).$$

Transferring the origin to $\left(-\frac{G}{A}, \frac{G^2-AC}{2F}\right)$, we get

$$Ax^2=-2Fy.$$

$$or \quad x^2=-\frac{2F}{A}y$$

which is the equation to a parabola.

Thus when A or B is zero, the equation represents a parabola.

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*It may be noted that when $k=0$, $\triangle=0$ and hence the equation represents a pair of straight lines.

But from relation (β), $ab=h^2$ when either A or B is zero.

Hence for $h^2=ab$ and $\Delta \neq 0$ the general equation of the second degree represents a parabola.

Case II. *Both A and B are of the same sign.*

Equation (2) on transforming the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$ becomes

$$Ax^2+By^2=K.$$

$$\text{or} \quad \frac{x^2}{\frac{K}{A}}+\frac{y^2}{\frac{K}{B}}=1$$

which is the equation to an ellipse real or imaginary according as k has or has not the same sign as A and B.

Thus when A and B both are of the same sign, the equation represents an ellipse. Also from the relation (β) $(ab-h^2)$ must be positive.

Hence for $ab>h^2$ and $\Delta \neq 0$ the general equation of the second degree represents an ellipse. Further if $A=B$, the ellipse becomes a circle.

Case III. *A and B are of opposite signs.*

As before the equation (2) becomes

$$\frac{x^2}{\frac{K}{A}}+\frac{y^2}{\frac{K}{B}}=1$$

which is the equation to a hyperbola, since $\frac{K}{A}$ and $\frac{K}{B}$ are of opposite signs (A and B being of opposite signs)

Thus for $ab<h^2$ and $\Delta \neq 0$ the general equation of the second degree represents a hyperbola.

Further the hyperbola will be rectangular if $A=-B$ or $A+B=0$ *i.e.* $a+b=0$ from relation (α).

Thus under all circumstances the general equation of the second degree represents a conic.

Ex. 1. What is represented by the following equations

(i) $7x^2-60xy+32y^2-106x+68y-37=0$

(ii) $5x^2+2xy+2y^2+24x+6y+29=0$

(iii) $2x^2+2y^2+5x+9y+18=0$

(iv) $6x^2-xy-12y^2-8x+29y-14=0$

(v) $16x^2-24xy+9y^2-60x-80y+400=0$

(vi) $x^2-y^2-10x+4y-7=0$

(vii) $4x^2+12xy+9y^2+16x+24y+16=0$

(viii) $4(x-2y+3)^2+9(2x+y-1)^2=80.$

Ex. 2. If d and d' be the perpendicular distances of a point P from two given lines not necessarily at right angles the locus of P is an ellipse or a hyperbola as $\frac{d^2}{\alpha^2}\pm\frac{d'^2}{\beta^2}=1$ where α^2 and β^2 are positive.

4·30. *To shew that a straight line intersects a conic in two points.*

Let the equation to the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0\quad\ldots\ldots\ldots(1)$$

and let

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r\quad\ldots\ldots\ldots(2)$$

be any straight line.

Then for the points of intersection of the conic and the straight line, substituting for x and y from (2) in (1), we get

$$a(x_1+r\cos\theta)^2+2h(x_1+r\cos\theta)(y_1+r\sin\theta)+b(y_1+r\sin\theta)^2+2g(x_1+r\cos\theta)+2f(y_1+r\sin\theta)+c=0.$$

or $$r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)+2r\{\cos\theta(ax_1+hy_1+g)+\sin\theta(hx_1+by_1+f)\}+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0\quad\ldots\ldots(3)$$

The above equation being a quadratic in r gives two values for the distance of the point of intersection from the point (x_1, y_1). Thus there are two points of intersection. Whether they are real and different, coincident or imaginary depends upon the nature of the roots of r.

Ex. 1. If P and Q be the points of intersection of a line passing through the point R (1, 2) and inclined at an angle $\tan^{-1}\frac{3}{4}$ to the axis of x with the conic $5x^2+6xy-5y^2-22x+18y-7=0$; shew that the rectangle $RP.RQ$ is equal to 100/107.

Ex. 2. Find the distance of the middle point of the points of intersection of the conic $2x^2+3y^2-4x-12y+13=0$ with the line $5x-12y+26=0$, from the point (2, 3).

4·31. *To find the equation to the tangent to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

at any point (x_1, y_1) *on it.*

If (x_1, y_1) is situated on the conic, the absolute term in equation (3) of the last article becomes zero and one value of r, therefore, vanishes. The equation then reduces to

$$r\,(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)$$
$$+2\{(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta\}=0$$

If the straight line is to be a tangent at (x_1, y_1) even this value of r should be zero, hence

$$(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta=0$$

Eliminating $\cos\theta$ and $\sin\theta$ between the above and (2) of the previous article, we get the required equation to the tangent, which is

$$(ax_1+hy_1+g)\,(x-x_1)+(hx_1+by_1+f)\,(y-y_1)=0$$

or $\quad axx_1+h\,(xy_1+yx_1)+byy_1+gx+fy$
$$-(ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1)=0$$

or $\quad axx_1+h(xy_1+yx_1)+byy_1$
$$+g(x+x_1)+f(y+y_1)+c=0. \quad (4)$$

Comparing the equation to the tangent with the equation to the conic, it will be seen that the former is obtainable from the latter by replacing therein, x^2 *by* xx_1, y^2 *by* yy_1, $2xy$ *by* xy_1+yx_1, $2x$ *by* $(x+x_1)$ *and* $2y$ *by* $(y+y_1)$.

Thus the tangent at (x_1, y_1) to the curve

(*i*) $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$.

(*ii*) $y^2=4ax$ is $yy_1=2a(x+x_1)$.

(*iii*) $\dfrac{x^2}{a^2}\pm\dfrac{y^2}{b^2}=1$ is $\dfrac{xx_1}{a^2}\pm\dfrac{yy_1}{b^2}=1$.

4·32. *To find the condition that the straight line* $lx+my+n=0$ *may touch the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let (x_1, y_1) be the point of contact.

Then $lx+my+n=0$,

and $(ax_1+hy_1+g)x+(hx_1+by_1+f)y+(gx_1+fy_1+c)=0$

represent the same straight line.

Hence $\dfrac{ax_1+hy_1+g}{l}=\dfrac{hx_1+by_1+f}{m}=\dfrac{gx_1+fy_1+c}{n}=\lambda$ (Say).

or
$$ax_1+hy_1+g-l\lambda=0$$
$$hx_1+by_1+f-m\lambda=0$$
$$gx_1+fy_1+c-n\lambda=0$$

Also $lx_1+my_1+n-0=0$, since (x_1, y_1) lies on the given line.

By eliminating (x_1, y_1), between the above equations the required condition comes out to be

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0.$$

or $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$,

where the capital letters denote the minors of the small letters in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

When the equations to the curves appear in their standard forms, the following conditions of tangency could be easily verified from the above :—

Curve	St. line	Condition of tangency
$y^2=4ax$	$lx+my+n=0$	$am^2=nl$
,,	$y=mx+c$	$c=a/m$
,,	$x \cos \alpha + y \sin \alpha = p$	$a \sin^2 \alpha + p \cos \alpha = 0$
$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$	$lx+my+n=0$	$a^2l^2 \pm b^2m^2=n^2$
,,	$y=mx+c$	$c^2=a^2m^2 \pm b^2$
,,	$x \cos \alpha + y \sin \alpha = p$	$a^2 \cos^2 \alpha \pm b^2 \sin^2 \alpha = p^2$.

Ex. 1. Determine the condition that the line represented by $y-13=m(x-2)$ may touch the conic $(3x-y+1)(x+y)=6$; and hence find the equations of the tangents to the given conic from the point $(2, 13)$.

Ex. 2. Find the equations to the tangents to the conic $3x^2+2xy+y^2-10x-14y+19=0$, which are inclined at an angle of $135°$ to the axis of x.

Ex. 3. Shew that the radical axis of the circles

$$x^2+y^2+8x+8y-11=0 \quad \text{and} \quad x^2+y^2+2x-4y-4=0$$

touches the conic $6x^2+18xy+27y^2+3x+3y-10=0$.

Ex. 4. Prove that the slopes of the tangents drawn from the origin to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c$$

are given by the quadratic equation

$$m^2(f^2-bc)+2fgm+g^2-ac=0$$

What do you deduce if $f^2=bc$?

4·40. *To shew that through any point (x_1, y_1) two tangents can be drawn to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If the straight line represented by (2) of Art. 4·30 is to be a tangent to the conic, the two roots of r in the equation (3) of the same article should coincide.

For this

$$\{(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta\}^2$$
$$=(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c) \quad ...(4)$$

or $$\{(ax_1+hy_1+g)+(hx_1+by_1+f)\tan\theta\}^2$$
$$=(a+2h\tan\theta+b\tan^2\theta)$$

$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c) \quad ....(5)$$

This equation, being a quadratic in $\tan\theta$ gives two values for θ. Thus there are two directions in which straight lines can be drawn from the point (x_1, y_1), to touch the given conic. In other words, through the point (x_1, y_1) two tangents can be drawn to the conic. Of course, whether they are real and different, coincident or imaginary depends upon the nature of the roots of the above equation.

4·41. *To find the equation to the pair of tangents that can be drawn from the point (x_1, y_1) to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=o.$$

Eliminating $\cos\theta$ and $\sin\theta$ between equation (2) of art. 4·30 and (4) above, we get the required equation, which

is $$\{(ax_1+hy_1+g)(x-x_1)+(hx_1+by_1+f)(y-y_1)\}^2$$
$$=\{a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2\}$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c) ...(6)$$

If $S \equiv ax^2+2hxy+by^2+2gx+2fy+c$

$S_1 \equiv ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$

and $T \equiv axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c$

Equation (6) can then be written as

$$(T-S_1)^2=S_1(S+S_1-2T)$$

$$\text{or} \qquad SS_1=T^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Thus the equation to the tangents drawn from the point (x_1, y_1) to the curve

(i) $x^2+y^2=a^2$ is

$$(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2,$$

(ii) $y^2=4ax$ is

$$(y^2-4ax)(y_1^2-4ax_1)=\{yy_1-2a(x+x_1)\}^2,$$

(iii) $\dfrac{x^2}{a^2}\pm\dfrac{y^2}{b^2}=1$ is

$$\{x^2/a^2\pm y^2/b^2-1\}\{x_1^2/a^2\pm y_1^2/b^2-1\}$$
$$=\{xx_1/a^2\pm yy_1/b^2-1\}^2$$

Ex. 1. Find the equation of the tangents from (1, 2) to the conic $2x^2-3xy-2y^2+x-y-2=0$, and find the angle between them.

Ex. 2. Tangents are drawn to the conic

$$ax^2+2hxy+by^2+2gx=0$$

from two points on the axis of y, which are equidistant from the origin. Prove that their points of intersection lie upon the straight line $hx+by=0$.

4·42. *To find the equation to the chord of contact of tangents drawn from the point (x_1, y_1) to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let the two points of contact be (p, q) and (r, s).

The equation of any straight line through these two points is

$$y-q=\frac{s-q}{r-p}(x-p) \qquad \ldots\ldots\ldots\ldots\ldots(a)$$

Now the tangents to the conic at (p, q) and (r, s) pass through (x_1, y_1). Hence

$$ax_1p + h(x_1q + y_1p) + by_1q$$
$$+ g(x_1 + p) + f(y_1 + q) + c = 0 \quad \ldots\ldots\ldots\ldots(b)$$

$$ax_1r + h(x_1s + y_1r) + by_1s$$
$$+ g(x_1 + r) + f(y_1 + s) + c = 0 \quad \ldots\ldots\ldots\ldots(c)$$

Subtracting (c) from (b)

$$(p - r)(ax_1 + hy_1 + g) + (q - s)(hx_1 + by_1 + f) = 0$$

$$\text{or} \quad \frac{s - q}{p - r} = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} \quad \ldots\ldots\ldots\ldots\ldots(d)$$

Substituting from (d) in (a)

$$(y - q)(hx_1 + by_1 + f) + (x - p)(ax_1 + hy_1 + g) = 0.$$

$$\text{or} \quad axx_1 + h(xy_1 + x_1y) + byy_1$$
$$+ g(x + x_1) + f(y + y_1) + c$$
$$= ax_1p + h(x_1q + y_1p) + by_1q$$
$$+ g(x_1 + p) + f(y_1 + q) + c$$
$$\equiv 0 \ldots\ldots\ldots\ldots\ldots[\text{from } (b)]$$

Thus the equation to the chord of contact of tangents drawn from the point (x_1, y_1) to the curve

(i) $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(ii) $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

(iii) $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} \pm \dfrac{yy_1}{b^2} = 1.$

4·43. *To find the locus of the points, tangents from which to the conic* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *are at right angles.*

The condition that the two tangents through (x_1, y_1) in Art. 4·41 may be at right angles is that in equation (7)

Co-efficient of $x^2 +$ co-efficient of $y^2 = 0$.

$$i.e. \quad as_1 - (ax_1 + hy_1 + g)^2 + bs_1 - (hx_1 + by_1 + f)^2 = 0.$$

$$or \quad (ab - h^2)(x_1^2 + y_1^2) + 2x_1(bg - fh) + 2y_1(af - gh) + c(a+b) - f^2 - g^2 = 0.$$

Whence the locus of (x_1, y_1) is

$$(ab - h^2)(x^2 + y^2) + 2x(bg - hf) + 2y(af - gh) + c(a+b) - f^2 - g^2 = 0. \quad \ldots(8)$$

This equation evidently represents a circle, which is called the **Director Circle** of the conic.

If the given conic happens to be a parabola *i.e.* if $h^2 = ab$, the above equation reduces to

$$2x(bg - hf) + 2y(af - gh) + c(a+b) - f^2 - g^2 = 0. \quad \ldots(9)$$

This equation, being of the first degree in x and y, represents a straight line, which as will be seen later is the directrix to the parabola represented by the general equation of the second degree.

Ex. Find the equation to the director circle of the conic

$$x^2 + xy + y^2 + x + y = 0.$$

4·50. *To find the equation to the chord of the conic* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, *having* (x_1, y_1) *for its middle point.*

The straight line given by equation (2) of Art. 4·30 will be the required chord if the two values of r in equation (3) of the same Art. be equal in magnitude but opposite in sign. For this the co-efficient of r must be zero, *i.e.*

$$(ax_1 + hy_1 + g) \cos\theta + (hx_1 + by_1 + f) \sin\theta = 0.$$

Eliminating $\cos\theta$ and $\sin\theta$ between the above equation and the equation under reference [(2) of Art. 4·30], we get

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0.$$

Using the abridged notation of Art. 4·41 this equation may be written as $$T = S_1*.$$

Thus the equation to this chord for

(i) $x^2+y^2=a^2$ is $xx_1+yy_1=x_1^2+y_1^2$

(ii) $y^2=4ax$ is $yy_1-2a(x+x_1)=y_1^2-4ax_1$

(iii) $x^2/a^2 \pm y^2/b^2=1$ is $xx_1/a^2 \pm yy_1/b^2=x_1^2/a^2+y_1^2/b^2$

Ex. 1. Shew that the locus of the middle points of all chords of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ which pass through the point (x', y') is the conic

$$ax^2+2hxy+by^2-x(ax'+hy'-g)-y(hx'+by'-f)-gx'-fy'=0.$$

Ex. 2. Find the inclination to the axis of x, and the length of that chord of the conic $2x^2+4xy+3y^2+5x-64y+127=0$, which is bisected at the point (1, 3).

4·51. *To find the co-ordinates of a point such that all chords of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

passing through that point are bisected there.

If (x_1, y_1) is to be the middle-point of all the chords of the conic, the equation

$$(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta$$

of the last article, must be satisfied for all values of θ. This is possible only when

$$\left.\begin{aligned} ax_1+hy_1+g=0 \\ hx_1+by_1+f=0 \end{aligned}\right\}.$$

Whence $$x_1=\frac{hf-bg}{ab-h^2}, \quad y_1=\frac{gh-af}{ab-h^2}.$$

Such a point is called the **Centre** of the conic.

The above co-ordinates for the centre lead to finite values except when $ab=h^2$. Hence conics have two classes

*It may as well be expressed by $T=T_1$.

viz. (*i*) those that have a finite centre, and (*ii*) those that have not. The former are called **Central Conics.** To this class belong the ellipses and the hyperbolas. To the other class belong the parabolas. Pairs of the straight lines and circles are only limiting forms of the above three general conics.

In the language of the differential calculus, if $f(x, y)=0$ represents a central conic, the co-ordinates of the centre are obtained by solving simultaneously

$$\frac{\delta f}{\delta x}=0 \quad \text{and} \quad \frac{\delta f}{\delta y}=0.$$

Ex. Find the centres of the following conics :—

(*i*) $x^2-xy+y^2-2x-2y=0$

(*ii*) $2x^2+3xy-2y^2-7x+y-2=0$

(*iii*) $10x^2-48xy-10y^2+38x+44y-5=0.$

4·52. *To find the locus of the middle points of a system of parallel chords of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

As in Art. 4·50, for (x_1, y_1) to be the middle point of the chord

$$(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta=0.$$

For a system of a parallel chords θ is constant. Let $\tan\theta$ be m. Then the required locus is

$$(ax+hy+g)+(hx+by+f)m=0$$

$$\text{or} \quad x(a+hm)+y(h+bm)+g+fm=0$$

The above equation being of the first degree in x and y represents a straight line. Also the co-ordinates of the centre will be seen to satisfy it. Hence it is a straight line passing through the centre. Such a straight line is called a **Diameter.** Every system of parallel chords has its diameter and vice-versa.

Ex. 1. Obtain the diameter of the chords of the conic $x^2+xy+y^2+2x=0$, which are parallel to the line $y=x$.

Ex. 2. Find the equation to that diameter of the conic $5x^2+6xy+5y^2+12x+4y+6=0$, which passes through the point (1, 2).

4·60. *To find the locus of the harmonic conjugates of a point* (x_1, y_1) *with respect to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

[If a straight line through the point O intersects a conic in P and Q, the point R is said to be the **Harmonic Conjugate** of O with respect to the conic, when $1/OP+1/OQ=2/OR$.]

If O be the point (x_1, y_1) and OP and OQ be the two values of r in equation (3) of Art. 4·30

$$\frac{1}{OP}+\frac{1}{OQ}=\frac{-2\{\cos\theta(ax_1+hy_1+g)+\sin\theta(hx_1+by_1+f)\}}{S_1}$$

$$=\frac{2}{OR}$$

or $S_1=-\{OR\cos\theta(ax_1+hy_1+g)+OR\sin\theta(hx_1+by_1+f)\}$

If R be the point (x, y), from equation (2) of Art. 4·30

$$OR\cos\theta=x-x_1$$

and $$OR\sin\theta=y-y_1$$

Hence the above equation becomes

$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=S_1$$

or $$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)$$
$$+f(y+y_1)+c=0.$$

Thus all points which are harmonic conjugates of a given point with respect to a conic lie on a straight line. This straight line is called the **Polar** of the given point with respect to the conic, and the given point is called its **Pole.**

If (x_2, y_2) be any point conjugate to the point (x_1, y_1) with respect to the conic, then from the above equation

$$ax_2x_1+h(x_2y_1+x_1y_2)+by_2y_1+g(x_2+x_1)+f(y_2+y_1)+c=0.$$

Since (x_1, y_1) and (x_2, y_2) in this equation are interchangeable the conjugate property is reciprocal *i.e.* if with respect to a given conic, R is the conjugate of O, then O is the conjugate of R. In other words this result may be stated thus.

If the polar of a point with respect to a conic passes through another point, the polar of the latter passes through the former.

It will be seen from Arts. 4·31, 4·42 and 4·60 that the form of the equation of the chord of contact of tangents drawn to the conic from the point (x_1, y_1), as also that of the polar of the point (x_1, y_1) with respect to the conic is the same as that of the equation to the tangent to the conic at the point (x_1, y_1), and can be written down by the same working rule.

Ex. Shew that the polar of any point, with respect to a conic, is parallel to the chord of the conic, which is bisected at that point.

4·61. *To find the co-ordinates of the pole of a given line with respect to the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Let the equation to the given line be

$$lx+my+n=0 \quad \dots\dots\dots\dots(i)$$

and let the co-ordinates of the required pole be (x_1, y_1).

The equation of the polar of this point with respect to the given conic is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+(gx_1+fy_1+c)=0 \;\ldots\ldots(ii)$$

Equations (i) and (ii) represent the same curve, hence

$$\frac{ax_1+hy_1+g}{l}=\frac{hx_1+by_1+f}{m}=\frac{gx_1+fy_1+c}{n} \;\ldots\ldots\ldots(iii)$$

Equations (iii) on solution will give us the values of x_1 and y_1 *i.e.* the co-ordinates of the pole.

It may be verified that if the pole of a line AB, with respect to a conic, lies on a line CD, then the pole of CD with respect to it lies on AB. Straight lines so related are called **Conjugate Lines.**

Ex. 1. Find the pole of the line

(i) $6x+9y+4=0$ with respect to $x^2+2xy+3y^2+2x+y+\frac{1}{2}=0$

(ii) $5x+y+4=0$,, $x^2+2xy-y^2+2x+1=0.$

Ex. 2. Prove that the line joining two points in the plane of a conic is the polar of the point of intersection of their polars with respect to the conic.

Ex. 3. Shew that the locus of the poles with respect to the central conic $ax^2+by^2=1$, of all tangents to the central conic $a'x^2+b'y^2=1$, is the conic $\dfrac{a^2}{a'}x^2+\dfrac{b^2}{b'}y^2=1$

Ex. 4. From each point on the line $x=4$, a perpendicular is drawn to its polar with respect to the ellipse $3x^2+4y^2=24$. Shew that the locus of the feet of the perpendiculars is the circle $x^2+y^2-3x+2=0.$

Miscellaneous Exercises.

1. Prove that the tangents to the ellipse $2x^2+3y^2=4$ and the hyperbola $3x^2-3y^2=1$ at their common points are at right angles to each other.

2. Shew that the conics $x^2-2xy+3y^2+4x-15y+18=0$ and $xy-4x-4y+14=0$ intersect at right angles at the point $(2, 3)$.

3. Shew that the ellipse $x^2/a^2+y^2/b^2=1$ and the parabola $ay^2+b^2(x-a)=0$ touch each other, and find the point of contact. Also shew that the tangents to the parabola at the other two points of intersection meet at the point $(2a, 0)$.

4. A pair of tangents to the conic $ax^2+by^2=1$ intercepts a constant length $2l$ on the x-axis. Prove that the locus of their point of intersection is the curve

$$by^2(ax^2+by^2-1)=al^2(by^2-1)^2.$$

5. Find the locus of the poles of the tangents of the conic $x^2/\alpha^2+y^2/\beta^2=1$ with respect to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

6. If a straight line touches a circle whose centre is the vertex of a parabola and whose diameter is equal to the latus-rectum, prove that the locus of its pole with respect to the parabola is a rectangular hyperbola.

7. If the polar of a point with respect to $x^2/a^2+y^2/b^2=1$ touches the hyperbola whose equation is $x^2/a^2-y^2/b^2=1$, the locus of the point is the hyperbola itself.

8. Shew that the locus of the pole of a line of constant length c which slides between the two axes with respect to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{a^4}{x^2}+\dfrac{b^4}{y^2}=c^2$.

9. Tangents are drawn from any point on the curve $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Shew that

the chord of contact will subtend a right angle at the centre of the ellipse.

10. Prove that the locus of the poles of tangents to the conic $ax^2+2hxy+by^2=1$ with regard to the conic $a'x^2+2h'xy+b'y^2=1$ is the conic
$a(h'x+b'y)^2-2h(a'x+h'y)(h'x+b'y)+b(a'x+h'y)^2=ab-h^2$.

11. A tangent to the parabola $(x+y)^2=4ax$ cuts the hyperbola $xy=c^2$ in the two points P and Q. Prove that the locus of the middle point of PQ is the hyperbola

$$2y(x-y)=ax.$$

12. Prove that the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which touch the hyperbola $x^2-y^2=a^2$ is $(x^2+y^2)^2=a^2(x^2-y^2)$.

13. Tangents are drawn from any point on the circle $x^2+y^2=a^2$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Prove that the locus of the middle points of the chords of contact is

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}^2=\frac{x^2+y^2}{a^2}$$

CHAPTER V.

PARABOLA.

5·1. We have seen in the last chapter that :

(1) The equation to a parabola having any point (h, k) for its focus and any straight line $Ax+By+C=0$ for its directrix is

$$(x-h)^2+(y-k)^2=\frac{(Ax+By+c)^2}{A^2+B^2} \quad \text{Art. 4·11}$$

(2) The necessary and sufficient conditions that the general equation of the second degree may represent a parabola are $\triangle \neq 0$ and $h^2=ab$.

(3) The standard or the simplest form of the equation to a parabola is $y^2=4ax$; Art. 4·20

where (*a*) latus rectum is $4a$, Art. 4·24

(*b*) the focal distance of any point (x', y') on it is $a+x'$, Art. 4·24

(*c*) the equation to the tangent at any point (x', y') is $yy'=2a(x+x')$. Art. 4·31

Hence the equation to the normal at any point (x', y') on it is $y-y'= -\frac{y'}{2a}(x-x')$.

(*d*) the condition that the st. line $y=mx+c$ may be a tangent is $c=a/m$. The corresponding conditions for the line $lx+my+n=0$ and $x \cos \alpha+y \sin \alpha=p$ are $nl=am^2$ and $p \cos \alpha+a \sin^2\alpha=0$, respectively. Art. 4·32

(e) the equation to the two tangents that can be drawn to it from any outside point (x', y') is $(y^2-4ax)(y'^2-4ax')=\{yy'-2a(x+x')\}^2$ Art. 4·41

(f) the equation to the chord of contact of the tangents drawn from any point (x_1, y_1) is $yy_1=2a(x+x_1)$. Art. 4·42

(g) the equation to the chord having (x_1, y_1) for its middle point is
$yy_1-2a(x+x_1)=y_1^2-4ax_1$ Art. 4·50

(h) the equation to the diameter of a system of parallel chords given by $y=mx+c$ is $y=2a/m$, which is a straight line parallel to the axis of the parabola. Art. 4·52

(i) the equation to the polar of any point (x_1, y_1) with respect to the parabola is $yy_1=2a(x+x_1)$. Art. 4·60

5·2. The co-ordinates of any point on the parabola $y^2=4ax$ in terms of a single variable 't' can be expressed as $(at^2, 2at)$, t being called the parameter for the point.

For brevity such a point is designated as the point 't'.

The equation to the chord joining the points 't_1' and 't_2' on the parabola $y^2=4ax$ is clearly

$$y-2at_1=\frac{2at_2-2at_1}{at_2^2-at_1^2}(x-at_1^2)$$

or $$y(t_1+t_2)=2(x+at_1t_2).$$

Hence the equation to the tangent to the parabola at the point 't' is

$$yt=x+at^2.$$

The point of intersection of the tangents at any points 't_1' and 't_2' could easily be verified to be

$$\{at_1t_2,\ a(t_1+t_2)\}.$$

Comparing the above equation to the tangent with the corresponding equation in the m—form* the geometrical significance of t is evident. It is seen to be $1/m$ *i.e.* the *Cotangent* of the angle which the tangent at the point makes with the axis of x.

Thus from this relation between 't' and 'm' the tangent $y=mx+\frac{a}{m}$ is seen to touch the parabola $y^2=4ax$ at the point $(a/m^2,\ 2a/m)$.

5·3. It has already been shown in Art. 4·40 that through any point (x_1, y_1) two tangents can be drawn to a conic. For the particular case of the parabola, this could be more easily demonstrated thus.

The straight line $ty=x+at^2$ is a tangent to the parabola $y^2=4ax$ for all values of t.

If it is to pass through a given point (x_1, y_1)

$$ty_1=x_1+at^2$$

$$or \quad at^2-ty_1+x_1=0$$

This equation being a quadratic in t gives two values for it. Hence two tangents to the parabola pass through (x_1, y_1). The nature of these tangents, however, depends upon the nature of the two values of 't' in the equation. They are real and different, coincident or imaginary according as

$$y_1^2 > = \quad or \quad < 4ax_1$$

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* $y=mx+\frac{a}{m}$ (See Art. 4·32)

i.e. according as the point lies without, upon or within the parabola.

Ex. 1. Shew that the ortho-centre of the triangle formed by three tangents of a parabola lies on the directrix.

Ex. 2. If two tangents to a parabola intercept a constant length on any fixed tangent, prove that the locus of their point of intersection is another equal parabola.

Ex. 3. The product of the tangents drawn from a point P to the parabola $y^2=4ax$ is equal to the product of the focal distance of P and the latus-rectum. Prove that the locus of P is the parabola $$y^2=4a(x+a)$$

5·40. From the equation to the tangent found in Art. 5·2 the equation to the normal to the parabola $y^2=4ax$ at the point 't' is seen to be

$$y-2at=-t(x-at^2)$$

$$\text{or} \quad y=-tx+2at+at^3$$

The corresponding equation in the m-form is obtained by putting $-t=m$ in the above. Thus the straight line

$$y=mx-2am-am^3$$

is a normal to the parabola at the point $(am^2, -2am)$.

5·41. That through any point (x_1, y_1) three normals can be drawn to a parabola could be shown with the help of any one of the above equations to the normals.

The straight line $y=mx-2am-am^3$ is a normal to the parabola for all values of m. If it is to pass through a fixed point (x_1, y_1)

$$y_1=mx_1-2am-am^3$$

$$\text{or} \quad am^3+m(2a-x_1)+y_1=0.$$

The above being a cubic equation in m gives three values of m, showing that three normals can be drawn through (x_1, y_1).

Ex. 1. The normals at the points where the line $y=mx+c$ cuts the parabola $y^2=4ax$ meet at P. Find the co-ordinates of the third point on the parabola the normal at which will also pass through P.

Ex. 2. Shew that the normal chord at the point on the parabola $y^2=4ax$ where the co-ordinates are equal, subtends a right angle at the focus.

Ex. 3. Shew that the locus of the point, from which the three normals to the parabola $y^2=4ax$ cut the axis, in the points whose distances from the vertex are in Arithmetical Progression, is $27ay^2=2(x-2a)^3$.

Ex. 4. From a variable point in a fixed normal two other normals are drawn to a parabola. Prove that the line joining their feet is parallel to a fixed line.

Ex. 5. The normals at P, Q and R are concurrent, and PQ meets the diameter through R on the directrix. Prove that PQ touches the parabola

$$y^2+16a(x+a)=0$$

Ex. 6. P, Q and R are the feet of the normals drawn to the parabola $y^2=4ax$ from the point (h, k). Shew that the sum of the squares of the sides of the triangle PQR is

$$2(h-2a)(h+10a).$$

Ex. 7. Normals are drawn to a parabola from a point P. Find the locus of P if one of the normals bisects the angle between the other two.

5·5. Below are enumerated some of the simple but important properties of the parabola, which are left as exercises for the student to verify :—

(*a*) The sub-tangent at any point of a parabola *i.e.* the portion of the x-axis intercepted between the tangent and the ordinate at the point, is bisected at the vertex, and is double the abscissa of the point.

(*b*) The sub-normal at any point of a parabola *i.e.* the portion of the x-axis intercepted between the, normal and the ordinate at the point, is a constant quantity, equal to half the latus-rectum.

(*c*) The tangent and normal at P

(*i*) make equal angles with its focal distance and the axis,

(*ii*) bisect the angles between its focal distance and the perpendicular from it upon the directrix,

(*iii*) make an intercept on the axis which is bisected at the focus.

(*d*) If the tangent at P meets the directrix in K, the angle PSK is a right angle.

(*e*) The locus of the foot of the perpendicular drawn from the focus to a tangent, is the tangent at the vertex.

(*f*) Tangents at right angles meet on the directrix.

(*g*) Tangents at the extremities of the focal chord meet at right angles on the directrix.

(*h*) Tangents from an external point subtend equal angles at the focus.

Miscellaneous Exercises.

1. Find the equation to the parabola whose focus is the point $(-1, 3)$ and whose vertex is the point $(4, 3)$. Also find the equation to the tangent parallel to the axis of y.

2. Find the equation to the parabola whose axis is the line $x+2=0$, whose vertex is the point $(-2, 3)$ and which passes through the point $(-1, 4)$.

3. From any point on the latus-rectum perpendiculars are drawn to the tangents at its extremities. Shew that the line joining the feet of these perpendiculars touches the parabola.

4. Two equal parabolas have their axes parallel and a common tangent at their vertices. Straight lines are drawn parallel to the direction of either axis. Shew that the locus of the middle points of the parts of the lines intercepted between the curves is an equal parabola.

5. Shew that the two parabolas which have the same focus and their axes in opposite directions are at right angles to each other.

6. The vertex of a triangle is fixed. The base is of constant length and moves along a fixed straight line Shew that the locus of the centre of its circumscribing circle is a parabola.

7. If the circle $x^2+y^2+ax+by+c=0$ cuts the parabola $y^2=4ax$ in four points, the algebraic sum of the ordinates of these points will be zero.

8. The triangle PQR is inscribed in the parabola $y^2=4ax$ and PQ, PR pass through the points $(0, 4a)$ $(0, -4a)$ respectively. Prove that QR touches the circle

$$x^2+y^2=4ax.$$

9. Prove that the locus of the point of intersection of normals at the extremities of a focal chord of a parabola is another parabola.

10. Circles are described on any two focal chords of a parabola as diameters. Prove that their common chord passes through the vertex of the parabola.

11. From any point on $y^2=a(x+m)$, tangents are drawn to $y^2=4ax$. Shew that the normals to the second parabola at the points of contact of these tangents intersect on a fixed straight line.

12. Tangents are drawn to the parabola $y^2=4ax$ from a given point P, which make angles θ_1, θ_2 to the axis of x. Find the locus of P when $\tan^2\theta_1+\tan^2\theta_2$ is constant.

13. A chord of the parabola $y^2=4ax$ passes through a fixed point (h, k). Through each extremity, a line is drawn parallel to the tangent at the other extremity. Prove that the locus of the point of intersection of these two lines is the parabola

$$2y^2-3ky=2a(x-3h).$$

14. Find the orthocentre of the triangle formed by joining the three points on a parabola the normals at which pass through a given point (h, k).

If the point (h, k) lies on the parabola itself, shew that the locus of the orthocentre is the parabola

$$y^2=a(x+6a).$$

15. Prove that the locus of the intersection of tangents to the parabola $y^2=4ax$ which intercept a fixed length on the directrix is

$$(x+a)^2(y^2-4ax)=l^2x^2.$$

16. P is the pole of the chord QQ', and perpendiculars from Q, P and Q' are drawn on any tangent to a parabola. Shew that the lengths of these perpendiculars are in Geometrical Progression.

17. Shew that the locus of the poles of the tangents to the parabola $y^2=2lx$ with respect to the circle $x^2+y^2=lx$

is a circle passing through the origin and touching the y-axis.

18. Shew that the locus of the poles of chords of the parabola $y^2=4ax$ which subtend a right angle at the focus is given by

$$x^2-y^2+4ax+4a^2=0.$$

19. Prove that the locus of the poles of chords of the parabola, $y^2=4ax$, which subtend a constant angle α at the vertex is the curve

$$(x+4a)^2=4(y^2-4ax)\cot^2\alpha.$$

20. If tangents be drawn to the parabola $y^2=4ax$ from any point on the parabola $y^2=4a'x$, the normals at the points of contact will meet on the curve

$$y^2(a-4a')^3+4aa'(x-2a)^3=0.$$

21. If two tangents to a parabola make equal angles with a fixed straight line, shew that the chord of contact must pass through a fixed point.

22. Shew that the locus of the poles of the normal chords of the parabola $y^2=4ax$ is

$$(x+2a)\,y^2+4a^3=0.$$

23. Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

24. The polar of P with respect to a parabola touches a circle whose centre is the focus, and whose diameter is equal to the latus-rectum. Find the locus of P.

[*Agra U. 1931*]

25. Two chords of a parabola are drawn in given directions so that their lengths are in a given ratio. Shew

that the locus of their point of intersection is a straight line. [*Agra U.* 1936]

26. Prove that the middle point of the intercept made on a tangent to a parabola by the tangents at two points P and Q lies on the tangent which is parallel to PQ. [*Agra U.* 1937]

27. A tangent to the parabola $y^2+4bx=0$ meets $y^2=4ax$ at P and Q. Prove that the locus of the middle point of PQ is $y^2(2a+b)=4a^2x$. [*All. U.* 1934]

28. Shew that the locus of the middle point of a variable chord of the parabola $y^2=4ax$, such that the focal distances of its extremities are in the ratio 2 : 1, is

$$9(y^2-2ax)^2=4a^2(2x-a)(4x+a). \qquad \text{[All. U. 1935]}$$

29. Two parabolas have the same axis, and tangents are drawn to one from points on the other. Prove that the locus of the middle points of the chords of contact is another fixed parabola.

30. Through each point of the straight line $x=my+h$ is drawn a chord of the parabola $y^2=4ax$ which is bisected in the point. Prove that the chord touches the parabola $(y+2am)^2=8a(x-h)$.

31. The locus of the middle points of the normal chords of the parabola $y^2=4ax$ is

$$\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a.$$

32. PV is the diameter through the fixed point P on a parabola, and QV any ordinate to it. QV is bisected in M and PV in N. Shew that the locus of the point of intersection of PM and QN is a parabola touching the given parabola.

CHAPTER VI.

ELLIPSE.

6·1. From Chapter IV it will be seen that :

(1) The equation to the ellipse having any point (h, k) for its focus and any straight line $Ax+By+C=0$ for its directrix is

$$(x-h)^2+(y-k)^2=e^2\ \frac{(Ax+By+C)^2}{A^2+B^2} \quad \text{Art. 4·11.}$$

where $e<1$.

(2) The necessary and sufficient conditions that the general equation of the second degree may represent an ellipse are

$$\triangle \neq 0 \quad \text{and} \quad h^2<ab.$$

(3) The standard or the simplest form of the equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1^*; \quad [b^2=a^2(1-e^2).] \qquad \text{Art. 4·21.}$$

where (*a*) latus rectum is $\dfrac{2b^2}{a}$ Art. 4·24.

(*b*) focal distances of any point (x', y') on it are $(a+ex')$ and $(a-ex')$. *Thus their sum is constant and equal to* $2a$. Art. 4·24.

(*c*) the equation to the tangent at any point (x', y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1. \qquad \text{Art. 4·31.}$$

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*In all that follows unless anything is stated to the contrary this shall be taken as the equation to the ellipse.

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Hence the equation to the normal at the same point is

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}}$$

$$i.e.\quad \frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2.$$

(*d*) the condition that the straight line $y=mx+c$ may be a tangent is $c^2=a^2m^2+b^2$. The corresponding conditions for the straight lines $lx+my+n=0$ and $x\cos\alpha+y\sin\alpha=p$ are $a^2l^2+b^2m^2=n^2$ and $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$ respectively. Art. 4·32.

(*e*) the equation to the two tangents that can be drawn to it from any outside point (x', y') is

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right\}$$
$$=\left\{\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right\}^2 \qquad \text{Art. 4·41.}$$

(*f*) the equation to the chord of contact of the tangents drawn from any point (x', y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1\,; \qquad \text{Art. 4·42.}$$

(*g*) the equation to the chord having (x_1, y_1) for its middle point is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \qquad \text{Art. 4·50.}$$

(h) the equation to the diameter of a system of parallel chords given by $y=mx+c$ is

$$\frac{x}{a^2}+\frac{my}{b^2}=0.$$

which is a straight line passing through the centre of the ellipse ; Art. 4·52.

(i) the equation to the polar of any point (x', y') with respect to the ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1. \qquad \text{Art. 4·60.}$$

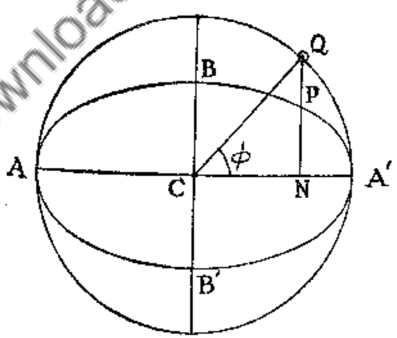
6·20. The co-ordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of a single variable 'ϕ' can be expressed as $(a \cos \phi, b \sin \phi)$.

For breirty such a point is designated as the point 'ϕ'. This variable 'ϕ' has an important geometrical significance :

A circle described on the major axis of the ellipse as diameter is called the **Auxiliary Circle** for that ellipse.

Let P be a point on the ellipse. Through P draw PN at right angles to the major axis and produce it back- wards to meet the auxiliary circle in Q. If the angle QCA' be denoted by 'ϕ', the co-ordinates of Q are obviously

$$(a \cos \phi, a \sin \phi)$$



Hence $CN = a \cos \phi$.

Also from the equation to the ellipse

$$\frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1$$

$$\text{or} \quad PN=b \sin \phi.$$

Thus P is the point $(a \cos \phi,\ b \sin \phi)$.

P and Q are called **Corresponding Points** and the angle ϕ is called the **Eccentric Angle*** of the point P.

6·21. The equation to the chord joining the points 'ϕ_1', and 'ϕ_2' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$y-b \sin \phi_1=\frac{b}{a}\ \frac{\sin \phi_2-\sin \phi_1}{\cos \phi_2-\cos \phi_1}(x-a \cos \phi_1)$$

$$\text{or} \quad \frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$$

Hence the equation to the tangent to the ellipse at the point 'ϕ' is

$$\frac{x \cos \phi}{a}+\frac{y \sin \phi}{b}=1. \quad \ldots\ldots\ldots\ldots(1)$$

Putting m for $-\frac{b}{a}\cot \phi$ in the above we get the corresponding equation in the m form *viz*

$$y=mx+\sqrt{a^2m^2+b^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The point of intersection of the tangents at the point 'ϕ_1' and 'ϕ_2' is

$$\frac{a \cos \frac{\phi_1+\phi_2}{2}}{\cos \frac{\phi_1-\phi_2}{2}},\quad \frac{b \sin \frac{\phi_1+\phi_2}{2}}{\cos \frac{\phi_1-\phi_2}{2}}$$

*It should be carefully noted that the eccentric angle of a point P is not the angle which CP makes with the major axis, but the angle which CQ (where Q is the point corresponding to P) makes with it.

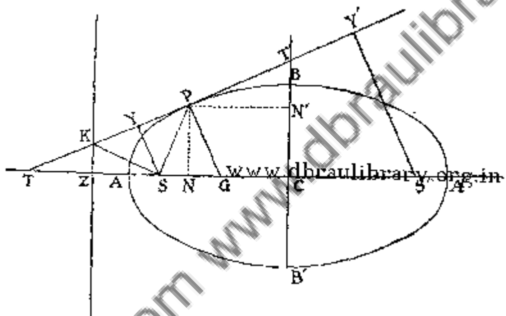
6·22. The equation to the normal at the point 'ϕ' can be easily shown to be

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2. \quad \text{............(1)}$$

Putting m for $\frac{a}{b} \tan \phi$ in the above, the corresponding *m* form of the equation will be seen to be

$$(mx - y)\sqrt{a^2 + b^2m^2} = m(a^2 - b^2). \quad \text{............(2)}$$

6·3. Some simple properties of the ellipse :—



(*a*) $SG = e.SP$ and $S'G = e\ S'P$. Also the tangent and the normal at P bisect the angles between its focal distances.

(*b*) If the tangent at P meets the directix in K the angle PSK is a right angle.

(*c*) The locus of the feet of the perpendiculars from the foci on any tangent is the auxiliary circle.

Also $\quad SY.S'Y' = b^2$.

(*d*) Tangents at right angles intersect on a fixed circle, called the **Director Circle.**

(*e*) Tangents at the exremities of a focal chord intersect on the directrix.

(*f*) If the tangent at P meets the major and minor axes in T and T', and N and N' be the feet of the perpendiculars from the point on these axes, then

(*i*) $CN.CT=a^2$

(*ii*) $CN'.CT'=b^2$.

6·40. It was shown in Art. 4·40 that through any point (x_1, y_1) two tangents can be drawn to a conic. For the particular case of the ellipse this could be more easily demonstrated with the help of Art. 6·21.

The straight line $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m. If it is to pass through a given point (x_1, y_1)

$$y_1=mx_1+\sqrt{a^2m^2+b^2}.$$

On simplification the above equation is seen to be a quadratic in m, giving two values for m. Hence two tangents to the ellipse pass through a given point.

The two values of m will be seen to be real and different, coincident or imaginary according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>=\quad or\quad <1.$$

i.e. according as the point (x_1, y_1) lies without, upon or within the ellipse.

6·41. *To shew that through a given point there can be drawn four normals to an ellipse.*

From Art. 6·22 the straight line

$$(mx-y)\sqrt{a^2+b^2m^2}=m(a^2-b^2)$$

is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m.

If this is to pass through a given point (x_1, y_1)

$$(mx_1 - y_1)\sqrt{a^2 + b^2m^2} = m(a^2 - b^2)$$

$$\text{or} \quad m^4b^2x_1^2 - 2m^3b^2x_1y_1 + m^2\{b^2y_1^2 + a^2x_1^2 - (a^2 - b^2)^2\} - 2ma^2x_1y_1 + a^2y_1^2 = 0.$$

The above being a fourth degree equation in m leads to four values for m, which proves the proposition.

6·52. Solved Example :—

Shew that if the normals at the four points 'ϕ_1', 'ϕ_2', 'ϕ_3' and 'ϕ_4' are concurrent, the sum of the eccentric angles of the points is an odd multiple of π.

If m_1, m_2, m_3 and m_4 be the roots of the fourth degree equation in m of the last article,

$$\Sigma m_1 = \frac{2y_1}{x_1}\,; \quad \Sigma m_1m_2 = \frac{b^2y_1^2 + a^2x_1^2 - (a^2 - b^2)^2}{b^2x_1^2}$$

$$\Sigma m_1m_2m_3 = \frac{2a^2y_1}{b^2x_1}$$

$$\text{and} \quad m_1m_2m_3m_4 = \frac{a^2y_1^2}{b^2x_1^2}.$$

Also it will be seen from Art 6·22 that

$$\tan\phi_r = \frac{b}{a}m_r, \quad \text{where} \quad r = 1, 2, 3 \text{ or } 4.$$

Now $\tan(\phi_1 + \phi_2 + \phi_3 + \phi_4)$

$$= \frac{\Sigma\tan\phi_1 - \Sigma\tan\phi_1\tan\phi_2\tan\phi_3}{1 - \Sigma\tan\phi_1\tan\phi_2 + \tan\phi_1\tan\phi_2\tan\phi_3\tan\phi_4}$$

$$= \frac{\dfrac{b}{a}\,\Sigma m_1 - \dfrac{b^3}{a^3}\,\Sigma m_1m_2m_3}{1 - \dfrac{b^2}{a^2}\,\Sigma m_1m_2 + \dfrac{b^4}{a^4}\,m_1m_2m_3m_4} = 0.$$

Hence $\phi_1 + \phi_2 + \phi_3 + \phi_4 = (2n+1)\pi.$

Examples.

1. Find the equation of the ellipse with axes parallel to the co-ordinate axes, centre at (5, 7), sum of the focal distances of any point 20 and one end of the minor axis at (5, 12). Find the co-ordinates of the focii.

2. Tangents are drawn at pairs of points of an ellipse whose eccentric angles differ by 2α; find the locus of their intersection.

3. A number of ellipses on the same major axis are cut by any common ordinate; Shew that the tangents at the intersections all meet in a point.

4. If the circle described on SS' as diameter cuts the minor axis in k and k' prove that the sum of the squares of the perpendiculars from k and k' on any tangent is constant.

5. Tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles θ_1, θ_2 with the major axis. Find the locus of their intersection when $\cot\theta_1+\cot\theta_2$ is constant.

6. Shew that the line $lx+my+n=0$ is a normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ when } \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

7. From C the centre of an ellipse a line CN is drawn perpendicular to PN, the normal at any point P on the ellipse, and PN is produced to T so that $PN.PT=ab$. Prove that the locus of T is a circle with centre C and radius $(a-b)$.

8. If the eccentric angles θ and ϕ of two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are connected by the relation

$\sec\theta + \sec\phi = 2$, prove that the chord through these points will touch the ellipse

$$\frac{4x^2}{a^2} + \frac{y^2}{b^2} = \frac{4x}{a}.$$

9. PQ is a focal chord of an ellipse. The tangents and normals at P and Q intersect in T and R respectively. Shew that TR passes through the other focus.

10. Tangents are drawn from points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the circle $x^2 + y^2 = r^2$. Shew that the chord of contact touches the ellipse $a^2x^2 + b^2y^2 = r^4$.

11. Find the locus of the intersection of tangents at the ends of chords of an ellipse, which are of constant length $2c$.

12. The polar of a point P with respect to an ellipse touches a fixed circle, whose centre is on the major axis and which passes through the centre of the ellipse. Shew that the locus of P is a parabola, whose latus-rectum is a third proportional to the diameter of the circle and the latus-rectum of the ellipse.

13. Find the length of the polar chord of the point (h, k) with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

14. Shew that the locus of the pole, of a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to its auxiliary circle, is a similar concentric ellipse, whose major axis is at right angles to that of the original ellipse.

15. Chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ always touch the concentric and coaxal ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. Find the locus of their poles.

16. Prove that the locus of the pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ when perpendicular on its polar from the centre is constant, is a concentric and coaxal ellipse.

17. Shew that the locus of the poles of the normal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

18. Shew that the lines $lx+my+n=0$ and $l'x+m'y+n'=0$ are conjugate lines with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, provided that $a^2ll'+b^2mm'=nn'$.

If two conjugate lines cut at right angles at (h, k) and one makes an angle θ with the axis of x, then

$$\tan 2\theta=\frac{2hk}{\{(h^2-a^2)^2+(k^2-b^2)^2\}}.$$

19. Shew that locus of the middle points of chords of an ellipse drawn through one end of the major axis is another ellipse. Find its centre.

20. Shew that the equation to the locus of the middle points of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which touch the circle $x^2+y^2=c^2$ is

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}^2=c^2\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}\right\}$$

21. The locus of the middle points of the chords of the conic
$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$
which touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

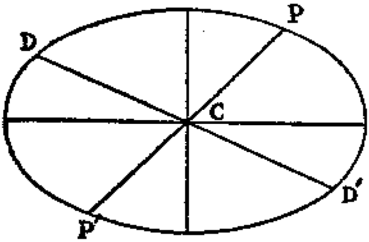
is the curve
$$\left\{\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}\right\}^2=\frac{a^2x^2}{(a^2+\lambda)^2}+\frac{b^2y^2}{(b^2+\lambda)^2}.$$

22. If the normals at the four points (x_1, y_1) ; (x_2, y_2) ; (x_3, y_3) and (x_4, y_4) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet in a point (h, k), prove that

(i) $$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$
$$=(y_1+y_2+y_3+y_4)\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)=4.$$

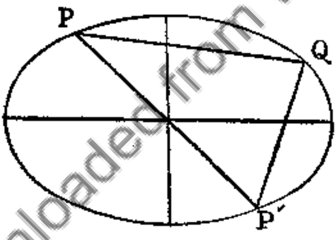
(ii) $\frac{x_1x_2x_3x_4}{h^2}$ and $\frac{y_1y_2y_3y_4}{k^2}$ are constants.

6·60. We have seen in Art. 6·1 (h) that the diameter of the system of parallel chords given by $y=mx+c$, to the ellipse $x^2/a^2+y^2/b^2=1$ is $x/a^2+my/b^2=0$. Denoting this diameter by $y=m_1x$, we can say that the diameter $y=m_1x$ bisects all chords parallel to the diameter $y=mx$, if
$$m_1=-b^2/a^2m$$
$$i.e,\quad mm_1=-b^2/a^2.$$



Since m and m_1 in the above result are interchangeable, $y=mx$ reciprocally bisects all chords parallel to $y=m_1x$. Two diameters PCP' and DCD' so related that each bisects chords parallel to the other are called **Conjugate Diameters.** The condition for the conjugacy of $y=mx$ and $y=m_1x$ is then $mm_1=-b^2/a^2$.

6·61. *To shew that if PCP' is a diameter and Q any point on the ellipse, the diameters parallel to the chords QP and QP', known as* **supplemental Chords,** *are conjugate.*



Let P and Q be the points 'ϕ' and 'θ'. Then the co-ordinates of P, P' and Q are

$$(a\cos\phi,\ b\sin\phi),$$
$$(-a\cos\phi,\ -b\sin\phi)$$
and $(a\cos\theta,\ b\sin\theta)$, respectively.

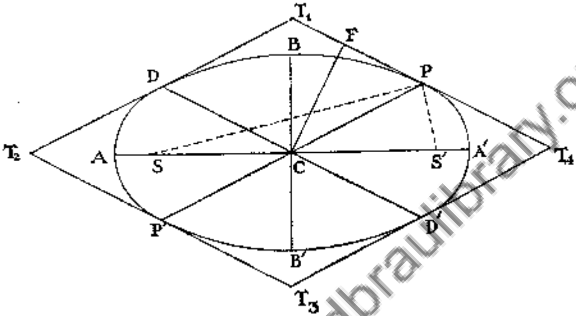
Hence the m's of QP and QP', and therefore, also of the diameters parallel to them, are

$$\frac{b}{a}\ \frac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi} \quad\text{and}\quad \frac{b}{a}\ \frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi}.$$

The product of these two m's being $-\dfrac{b^2}{a^2}$, the diameters are conjugate.

6·62. Some properties of conjugate diameters.

(a) *The eccentric angles of conjugate diameters differ by $\pi/2$.*



If PCP' and DCD' be two conjugate diameters of an ellipse, P and D being the points θ and ϕ respectively, then tangents at P and D are

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$$\text{and} \quad \frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 1$$

Parallel diameters are

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 0$$

$$\text{and} \quad \frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 0$$

Condition of conjugacy gives

$$\cos \phi \cos \theta + \sin \phi \sin \theta = 0$$

$$\text{or} \quad \cos (\phi - \theta) = 0$$

$$\text{or} \quad \phi - \theta = \pi/2^*.$$

*$(\phi - \theta)$ should not be confused with angle DCP.

(b) *Tangents at the extremities of a diameter are parallel to its conjugate diameter, and therefore to the system of chords bisected by the former.*

Now m of the tangent at P is $-\dfrac{b\cos\theta}{a\sin\theta}$.

Also m of DCD' is $\dfrac{b\sin\phi}{a\cos\phi}$

By (a) the two m's are the same.

(c) $CP^2+CD^2=a^2+b^2$.

The co-ordinates of P and D from (a) above are $(a\cos\theta,\ b\sin\theta)$ and $(-a\sin\theta,\ b\cos\theta)$

Hence

$$CP^2+CD^2=(a^2\cos^2\theta+b^2\sin^2\theta)+(a^2\sin^2\theta+b^2\cos^2\theta)=a^2+b^2.$$

(d) *The area of the parallelogram* $T_1T_2T_3T_4$ is $4ab$.

$T_1T_2T_3T_4=4\,T_1PCD.$

$=4\,CD.\,CF$ where F is the foot of the perpendicular from C on PT_1.

Now $CD=\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}$

Equation to PT_1 is $bx\cos\theta+ay\sin\theta-ab=0$

$$\text{and so } CF=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

$$=\frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}$$

$$\left[\because \phi-\theta=\frac{\pi}{2}\right].$$

Hence $CD.\,CF=ab$

So that $T_1T_2T_3T_4=4ab.$

(e) $$SP.\,S'P = CD^2$$

$$SP = a + ae\cos\theta$$

$$S'P = a - ae\cos\theta$$

Therefore $$SP.\,S'P = a^2 - a^2e^2\cos^2\theta$$
$$= a^2 - (a^2 - b^2)\cos^2\theta$$
$$= a^2\sin^2\theta + b^2\cos^2\theta$$
$$= CD^2.$$

(f) *There is one, and only one, set of* **equi-conjugate** *diameters i.e. conjugate diameters equal in length.*

If $CP^2 = CD^2$

$$a^2\cos^2\theta + b^2\sin^2\theta = a^2\sin^2\theta + b^2\cos^2\theta$$

or $$(a^2 - b^2)\cos 2\theta = 0$$

or $$\theta = \pi/4 \quad [\because a \neq b].$$

Thus P and D are the points $(a/\sqrt{2}, b/\sqrt{2})$ and $(-a/\sqrt{2}, b/\sqrt{2})$ and so the equations to equi-conjugate diameters are $ay = bx$ and $ay + bx = 0$ *i.e.* they are the diagonals of the rectangle formed by the tangents to the ellipse at the extremities of its axes.

Each one of these equi-conjugate diameters is of length $\sqrt{2(a^2+b^2)}$

Also the angle between them is $\tan^{-1}\dfrac{2ab}{a^2+b^2}$

Ex. 1. Prove that the polar of a point P with respect to an ellipse is parallel to the diameter which is conjugate to the diameter through P.

Ex. 2. Find the equation to the diameter of the ellipse $x^2/a^2 + y^2/b^2 = 1$ conjugate to $lx + my = 0$.

Ex. 3. Prove that the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points whose eccentric angles are θ and $90° - \theta$, meet on one of the equi-conjugate diameters.

Ex. 4. CP and CD are the conjugate diameters of an ellipse. Find the locus of middle point of PD.

Ex. 5. Through the foci of an ellipse, perpendiculars are drawn to a pair of conjugate diameters. Prove that they meet on a fixed concentric ellipse.

Ex. 6. If the points of intersection of the ellipses $x^2/a^2+y^2/b^2=1$ and $x^2/\alpha^2+y^2/\beta^2=1$ be at the extremities of conjugate diameters of the former, prove that $a^2/\alpha^2+b^2/\beta^2=2$.

Ex. 7. Prove that the locus of the intersections of the tangents at the extremities of a pair of conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$ is the ellipse $x^2/a^2+y^2/b^2=2$.

Miscellaneous Exercises.

1. A circle is described which passes through the point (0, 2) and touches the circle $x^2+y^2=16$ internally. Prove that the locus of its centre is an ellipse whose foci are the origin, and the point (0, 2).

2. Find the common tangents to the ellipses $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2+b^2}=1$; and shew that they are parallel to the diagonals of the rectangle whose sides are their directrices.

3. A parabola is drawn to touch the major axis of an ellipse, the point of contact being the centre of the ellipse and the vertex of the parabola. Find the ratio of the axes of the ellipse if the curves cut at right angles.

4. Prove that the locus of the intersection of the normals at the ends of conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is the curve

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$$

5. QQ' is any chord of an ellipse parallel to one of its equi-conjugate diameters, and the tangents at Q and Q'

meet in P. Shew that the circle QPQ' passes through the centre of the ellipse.

6. The tangents at P and Q on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ meet at the point T, and the lines PS and QS' meet on the curve. Shew that the locus of T is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left\{\frac{1-e^2}{1+e^2}\right\}^2=1.$$

7. Lines are drawn through the origin perpendicular to the tangents from a point P to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Find the locus of P if the lines are conjugate diameters of the ellipse.

8. Shew that if a polar to an ellipse touches the circle described on the semi-minor axis as diameter, the pole will lie on a parabola whose vertex is at the end of the minor axis.

9. If (h, k) be the middle point of a chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, prove that the equation to the circle described on the chord as diameter is

$$\left\{\frac{h^2}{a^2}+\frac{k^2}{b^2}\right\}\left\{(x-h)^2+(y-k)^2\right\}$$
$$=a^2b^2\left\{\frac{h^2}{a^4}+\frac{k^2}{b^4}\right\}\left\{1-\frac{h^2}{a^2}-\frac{k^2}{b^2}\right\}.$$

10. Find the locus of the middle points of the chords of contact of tangents which cut at right angles.

11. Find the locus of the middle points of chords of an ellipse which subtend a right angle at the centre.

12. Shew that the locus of the middle points of the chords of constant length $2c$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}\right\}+\frac{c^2}{a^2b^2}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}=0.$$

13. Shew that if (x', y') be the middle point of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (ξ, η) the point of intersection of the normals, and (x, y) that of the tangents at its extremities, then

$$\frac{a^2\xi}{x}+\frac{b^2\eta}{y}=(a^2-b^2)\left\{\frac{xx'}{a^2}-\frac{yy'}{b^2}\right\}.$$

14. If P is any point on an ellipse, and F a focus, shew that the circle on FP as diameter touches the auxiliary circle.

15. Find the locus of a point P, such that the tangents drawn from it to an ellipse, and their chord of contact, form a triangle whose centroid lies on the curve.

16. Prove that if the normals at four points of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, and two of the points lie on the line $\frac{lx}{a}+\frac{my}{b}+1=0$, the other two will lie on the line $\frac{x}{al}+\frac{y}{bm}=1$.

Also prove that if two lines drawn through the point $(2a/3, 2b/3)$ meet the ellipse at four points, the normals at which are concurrent, one of the lines will be $4bx-ay=2ab$, and find the equation of the other.

17. Shew that if a circle cuts the ellipse in four points, the sum of their eccentric angles is an even multiple of π.

18. The normals at the points P, Q, R, T on an ellipse meet in a point, and the circle through P, Q, R cuts the conic again in T'. Shew that TT' is a diameter of the ellipse.

19. Shew that the length of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which touches the ellipse $\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1$ is $\frac{2\lambda b_1{}^2}{ab}$, where b_1 is the semi-diameter parallel to the chord.

20. The polars of P_1 (x_1, y_1) and P_2 (x_2, y_2) with respect to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the curve in Q_1R_1 and Q_2R_2 respectively ; shew that the six points lie on the conic

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}\left\{\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\right\}$$
$$=\left\{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right\}\left\{\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right\}$$

21. PQ is a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, such that the lines joining it to the foci meet on the ellipse. Prove that the locus of the pole of the chord is

$$\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1.$$

22. If the normals at the points where the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is intersected by the lines $lx+my=1$ and $l'x+m'y=1$ meet in a point, shew that $a^2ll'=b^2mm'=-1$.

23. Find the equations to the two parabolas passing through the feet of the normals drawn from the point (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

CHAPTER VII.

HYPERBOLA.

7·1. As for the parabola and the ellipse, it will be seen from Chapter IV for the hyperbola also, that :

(1) The equation to the hyperbola, having any point (h, k) for its focus and any straight line $Ax+By+C=0$ for its directrix, is

$$(x-h)^2+(y-k)^2=e^2\frac{(Ax+By+C)^2}{A^2+B^2}, \quad \text{Art. 4·11.}$$

where $e>1$

(2) The necessary and sufficient conditions, that the general equation of the second degree may represent a hyperbola are

$$\Delta \neq 0 \quad \text{and} \quad h^2>ab$$

(3) The standard or the simplest form of the equation to the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1^*; \quad [b^2=a^2(e^2-1)] \qquad \text{Art. 4·21.}$$

where (*a*) latus-rectum is $\dfrac{2b^2}{a}$ Art. 4·24.

(*b*) focal distances of any point (x', y') on it are $ex'+a$ and $ex'-a$. *Thus their difference is constant and equal to* $2a$. Art. 4·24.

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*In all that follows, unless any thing is stated to the contrary this shall be taken as the equation to the hyperbola.

(c) The equation to the tangent at any point (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \qquad \text{Art. 4·31.}$$

Hence equation to the normal at the same point is

$$\frac{x-x'}{\frac{x'}{a^2}}+\frac{y-y'}{\frac{y'}{b^2}}=0$$

$$i.e. \quad \frac{a^2x}{x'}+\frac{b^2y}{y'}=a^2+b^2.$$

(d) the condition that the straight line $y=mx+c$ may be a tangent, is $c^2=a^2m^2-b^2$. The corresponding conditions for the straight lines $lx+my+n=0$ and $x\cos\alpha+y\sin\alpha=p$ are $a^2l^2-b^2m^2=n^2$ and $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$ respectively. Art. 4·32.

(e) the equation to the two tangents that can be drawn to it from any outside point (x', y') is

$$\left\{\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right\}\left\{\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right\}$$

$$=\left\{\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right\}^2 \qquad \text{Art. 4·41.}$$

(f) the equation to the chord of contact of tangents drawn from the point (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \qquad \text{Art. 4·42.}$$

(g) the equation to the chord having (x_1, y_1) for its middle point is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}. \qquad \text{Art. 4·50.}$$

(h) the equation to the diameter of a system of parallel chords given by $y = mx + c$ is

$$\frac{x}{a^2} - \frac{my}{b^2} = 0$$

which is a straight line passing through the centre. Art. 4·52.

(i) the equation to the polar of any point (x', y') with respect to the hyperbola is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \qquad \text{Art. 4·60.}$$

7·20. The co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of a single variable 'ϕ' can be expressed as $(a \sec \phi, b \tan \phi)$. Such a point is designated as the point 'ϕ'. Unlike the ellipse, however, we do not propose to investigate the geometrical significance of this variable ϕ for the hyperbola.

7·21. The equation to the tangent at 'ϕ' will be easily seen to be

$$\frac{x \sec \phi}{a} - \frac{y \tan \phi}{b} = 1.$$

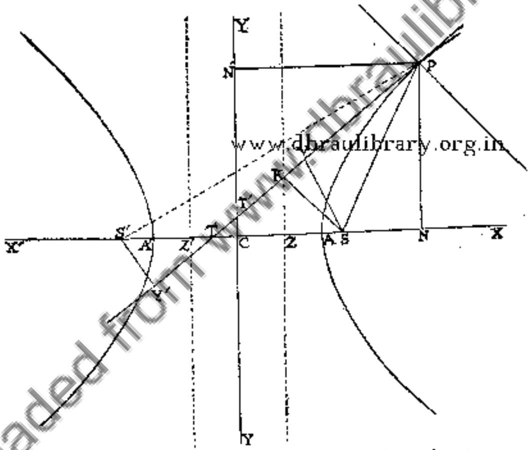
7·22. The equation to the normal at 'ϕ' will be

$$ax \cos \phi + by \cot \phi = a^2 + b^2.$$

7·23. The equation to the tangent and the normal in the m-form will be seen to be the same as the corresponding equations in the case of the ellipse with the difference that b^2 will be replaced by $-b^2$.

Arguing as in the case of the ellipse it could also be shown that through any given point two tangents and four normals can be drawn to a hyperbola.

7·3. The properties of the ellipse enumerated in Art. 6·3 can be shown to be true for the hyperbola as well. The proofs are left as exercises to the student. Thus



(a) $SG = e.SP$ and $S'G = e.S'P$. Also the tangent and normal at P bisect the angles between the focal distances.

(b) If the tangent at P meets the directrix in K, the angle KSP is a right angle.

(c) The locus of the feet of the perpendiculars from the foci on any tangent is the auxiliary circle i.e. the circle on the transverse axis as diameter. Also $SV.\ S'V' = -b^2$.

(*d*) Tangents at right angles intersect on a fixed circle, called the **Director Circle** of the hyperbola.

(*e*) Tangents at the extremities of a focal chord intersect on the directrix.

(*f*) If the tangent at P meets the transverse and conjugate axes in T and T' respectively and N and N' be the feet of the perpendiculars from the point on these axes

(*i*) $CN.CT=a^2$

(*ii*) $CN'.CT'= -b^2$.

Examples.

1. Find the equation to the hyperbola whose directrix is $2x+y=1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

2. Find the eccentricity of a hyperbola whose transverse axis is $2a$, whose axes are the co-ordinate axes and which passes through the point (x', y').

3. The difference of focal distances of any point on a hyperbola is 8, and the distance between its focus and directrix is $2/\sqrt{5}$. Find its equation, if its principal axes are parallel to the co-ordinate axes and its centre is at the point $(3, -2)$.

Also find its conjugate axis, latus-rectum and the vertices.

4. Find the equation to an equilateral hyperbola having its centre at the point $(4, 5)$, with its axes parallel

to the co-ordinate axes and with one of its vertices on the $y-$ axis.

Find also the equations to the directrices and the co-ordinates of the foci.

5. If a variable line forms with two fixed perpendicular lines a triangle of constant area, shew that the locus of a point which divides the intercept made on the variable line in a given ratio is a hyperbola.

6. Tangents are drawn to the parabola $y^2=4ax$ making a constant angle α with each other. Prove that the locus of their intersection is a hyperbola whose eccentricity is $sec\ \alpha$.

7. Given a chord of a parabola and the direction of its axis, shew that the locus of the focus is a hyperbola whose foci are at the extremities of the given chord.

8. The tangent at a point P of an ellipse cuts the hyperbola having the same axes in Q and R. If C is the common centre of the two curves and V the middle point of QR, prove that the angle VCP is bisected internally and externally by the axes.

9. Find the condition that the line $y=mx+c$ may touch both the parabola $y^2=4px$ and the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

10. Tangents are drawn to the hyperbola from the point (h, k), and make angles θ and θ' with the axis of x

Prove that $\quad tan\ \theta+tan\ \theta'=\frac{2hk}{h^2-a^2}$.

11. Prove that the locus of the point of intersection of tangents to the curve $x^2 - y^2 = a^2$ which are inclined to each other at an angle of 45°, is the curve

$$(x^2+y^2)^2 = 4a^2(a^2+y^2-x^2).$$

12. Through one of the vertices A and the extremities P, P' of a double ordinate of a hyperbola, a circle is drawn cutting the axis again in K. If G be the foot of the normal at P, prove that GK is of constant length.

13. Shew that in a hyperbola the circle on the line joining the focus to any point on it touches the auxiliary circle.

14. Shew that the condition that the line $lx+my+n=0$ may be a normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{is} \quad \frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}.$$

15. A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N and the rectangle $MCNP$ is completed. Shew that the locus of P is a hyperbola.

16. Find the locus of the point of intersection of two normals to a hyperbola which cut at right angles.

17. Find the locus of the poles of chords of a hyperbola which subtend a right angle at its centre.

18. Prove that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is

$$(y^2-x^2)^3 = 4a^2x^2y^2.$$

19. Find the condition that a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ subtends a right angle at the centre.

20. From points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$. Prove that the locus of the middle points of the chords of contact is the curve

$$(x^2-y^2)^2=a^2(x^2+y^2).$$

7·40. It has been shown in Art. 4·30 that a straight line intersects a conic in two points. When both of these points of intersection are situated at infinity, the straight line is called an **'Asymptote'** of the conic.

That there are such straight lines associated with some of the conics can be shown thus :

Let a given curve of the second degree be represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ and let $y=mx+k$ represent a given straight line.

The abscissae of the points of intersection of the straight line and the curve are given by

$$ax^2+2hx(mx+k)+b(mx+k)^2+2gx+2f(mx+k)+c=0.$$

By the theory of quadratic equations, the roots of this equation are infinite if the coefficients of x^2 and x respectively vanish *i.e.* if

$$a+2hm+bm^2=0 \quad\ldots\ldots\ldots\ldots(1)$$

$$\text{and}\quad kh+bmk+g+fm=0. \quad\ldots\ldots\ldots\ldots(2)$$

Now equation (1) will give two real values for m if $h^2>ab$.

Also for $h^2=ab$, it will give $m=-h/b$. This value of m when substituted in (2) fails to give any finite value for k.

Hence real finite values for m and k are possible when, and only when, $h^2 > ab$. Thus *asymptotes* are real and finite straight lines in the case of a hyperbola, and no such lines are associated with the parabola or the ellipse.

7·41. *To find the asymptotes of the hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

The given hyperbola is intersected by the straight line $y = mx + c$ in points whose abscissae are given by

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

$$\text{or} \qquad x^2\left\{\frac{1}{a^2} - \frac{m^2}{b^2}\right\} - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

In order that the straight line may become an asymptote

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \quad \text{or} \quad m = \pm\frac{b}{a}.$$

$$\text{and} \qquad \frac{cm}{b^2} = 0 \quad \text{or} \quad c = 0.$$

Hence the required asymptotes are

$$\frac{x}{a} + \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} = 0$$

$$\text{or} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

7·42. What has been said with regard to the asymptotes so far, must have left the student wondering as to the physical significance of these straight lines. To clarify this notion let us consider one of them, say

$$\frac{x}{a} - \frac{y}{b} = 0.$$

The length of the perpendicular from any point (x_1, y_1) on the hyperbola upon this asymptote is

$$\frac{\dfrac{x_1}{a}-\dfrac{y_1}{b}}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}}} \quad \text{where} \quad \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1.$$

$$=\frac{b\{x_1-\sqrt{x_1^2-a^2}\}}{\sqrt{a^2+b^2}}$$

$$=\frac{b}{\sqrt{a^2+b^2}}\left\{x_1-x_1\sqrt{1-\frac{a^2}{x_1^2}}\right\}.$$

$$=\frac{b}{\sqrt{a^2+b^2}}\,\frac{a^2}{2x_1} \quad [\text{for large values of } x_1]$$

As x_1 increases the length of this perpendicular decreases *i.e. as we move further and further away from the origin along the axis of x, the curve and the asymptote come nearer and nearer to each other.*

Also if we have a straight line inclined to the axis of x at a smaller angle than $\frac{x}{a}-\frac{y}{b}=0$, say, the straight line $\frac{y}{b}=\frac{x}{a}(1-\xi)$ where ξ is positive, it will meet the hyperbola in points whose abscissae are given by

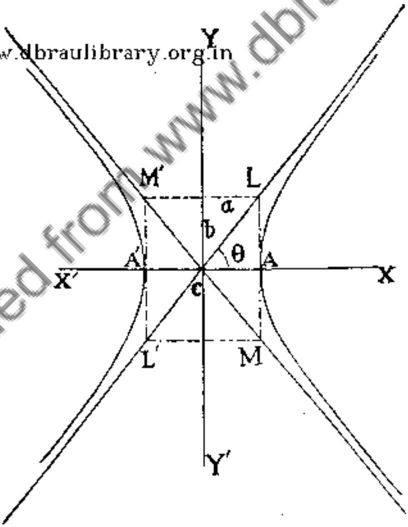
$$\frac{x^2}{a^2}-\frac{x^2}{a^2}(1-\xi)^2=1$$

$$\textit{or} \quad \frac{x^2}{a^2}\{1-1+2\xi-\xi^2\}=1$$

$$\textit{or} \quad x^2=\frac{a^2}{\xi(2-\xi)}$$

The two abscissae are equal in magnitude but opposite in sign. They increase in magnitude as ξ decreases, tending to infinity as ξ tends to zero. *Thus of the two points at infinity, in which the asymptote* $\frac{x}{a}-\frac{y}{b}=0$ *intersects the hyperbola, one is situated on the positive side of the axis of* x *and the other on its negative side.*

The case of the asymptote $\frac{x}{a}+\frac{y}{b}=0$ could be similarly argued, and the asymptotes are as shown in the figure below :



A geometrical construction for the asymptotes is apparent. They are the diagonals of the rectangle formed by the straight lines

$$x=\pm a, \quad y=\pm b.$$

7·43. If 2θ be the angle between the asymptotes

$$tan\ \theta = b/a.$$

Hence if the conjugate and transverse axes of a hyperbola are equal, $tan\ \theta = 1$ *i.e.* $\theta = 45°$, and the asymptotes are at right angles. This is why such a hyperbola is not only called equilateral but *rectangular* as well. (See foot-note p. 58.)

7·44. *Since asymptotes intersect the curve at two points at infinity, they can be said to be the tangents at these points, and their equations can be derived as such.*

The equation to the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at any point (x', y') is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

$$\text{or} \quad \frac{xx'}{a^2} \pm \frac{y}{b}\sqrt{\frac{x'^2}{a^2} - 1} = 1 \quad \left\{ \because \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \right\}.$$

$$\text{or} \quad \frac{x}{a^2} \pm \frac{y}{b}\sqrt{\frac{1}{a^2} - \frac{1}{x'^2}} = \frac{1}{x'}$$

$$\text{or} \quad \frac{x}{a} \pm \frac{y}{b} = 0 \qquad \text{as } x' \text{ tends to infinity.}$$

7·45. The equation to the pair of tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from any point (x', y') is

$$\left\{ \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \right\} \left\{ \frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1 \right\} = \left\{ \frac{xx'}{a^2} - \frac{yy'}{b^2} - 1 \right\}^2$$

If this point (x', y') be the centre of the hyperbola *i.e.*, the point $(0, 0)$ the above equation reduces to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

Thus the asymptotes to a hyperbola may also be looked upon as tangents drawn to it from its centre.

7·46. The above definition of the asymptotes furnishes a very handy method for finding the asymptotes of the curve given by the general equation *viz.*,

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Looking upon these asymptotes as the tangents to the curve from its centre (x_1, y_1) the required equation is seen to be

$$(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)$$
$$=\{axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c\}^2.$$

or $ax^2+2hxy+by^2+2gx+2fy+c$
$$=x_1(ax_1+hy_1+g)+y_1(hx_1+by_1+f)+gx_1+fy_1+c$$
$$=gx_1+fy_1+c=\frac{\Delta}{ab-h^2} \qquad \text{Arts. 4·51 and 2·86.}$$

It has been shown in Art 2·87 that the two straight lines represented by the above equation are parallel to the straight lines given by $ax^2+2hxy+by^2=0$. Hence if the second degree terms in the equations to two conics are the same, *i.e.* if their equations differ only in first degree and constant terms, the two conics will have their asymptotes in the same directions.

7·47. It will be seen in the above equation to the asymptotes, as should have been noted in Art. 7·42 also, that the equation to the asymptotes to a given conic differs from the equation to the conic only in its absolute term. This provides another handy method for getting the equation to the asymptotes. To the given equation to the conic, we have only to add a constant such that the new equation may represent a pair of straight lines.

Thus the equation to the asymptotes to the general conic may be written as

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0. \quad \text{.........}(i)$$

where λ is determined by

$$ab(c+\lambda)+2fgh-af^2-bg^2-(c+\lambda)h^2=0.$$

$$i.e. \quad \lambda= -\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}= -\frac{\Delta}{ab-h^2}. \quad \text{...}(ii)$$

Substituting the value of λ from (ii) in (i) above, the equation to the asymptotes is seen to be

$$ax^2+2hxy+by^2+2gx+2fy+\frac{2fgh-af^2-bg^2}{ab-h^2}=0.$$

This equation is independent of c which shews that the constant term in the equation to the conic has no bearing upon the equation to its asymptotes. Hence all conics whose equations differ only in constant terms have the same asymptotes.

Exercises.

1. Find the equations to the asymptotes of the following hyperbolas :—

   (i) $xy-3x+4y=0.$

   (ii) $5x^2-3y^2-2xy-4x+4y-4=0.$

   (iii) $2x^2-3y^2+xy-4x-y-10=0.$

2. Find the equation of the hyperbola whose asymptotes are the straight lines $3y-4x-12=0$ and $4x+3y-12=0$ and which passes through the origin.

3. Find the equation of the hyperbola with centre at (1, 2), asymptotes parallel to $2x+3y=0$ and $2y-x=0$, and passing through the point (5, 3).

4. A hyperbola and an ellipse, whose equations are respectively $x^2-7y^2=14$ and $9x^2+25y^2=225$, intersect at a point P. The ordinate of P is produced to meet the auxiliary circle of the ellipse in Q. Prove that Q lies on one of the asymptotes of the hyperbola.

5. Shew that in a rectangular hyperbola the straight line joining the centre with the pole makes the same angle with the transverse axis as the straight line drawn from the centre perpendicular to the polar; and that the semi-axis is a mean proportional between the lengths of these straight lines.

6. The normal at P_1 to the rectangular hyperbola $xy=c^2$ meets the curve again at P_2. The normal at P_2 meets the curve again at P_3 and so on. Prove that if y_1, y_2, y_3.........are the ordinates of these points

$$y_1^3y_2=y_2^3y_3=\ldots\ldots\ldots=-c^4$$

7. PQ, PR are two chords of an equilateral hyperbola, and the angle at P is a right angle. Prove that the perpendicular from P to QR touches the hyperbola.

7·50. Like the ellipse the hyperbola has its pairs of conjugate diameters, the condition of conjugacy of Art. 6·60 becoming $mm_1=b^2/a^2$.

The asymptotes are thus seen to be self-conjugate diameters.

7·51. Any diameter $y=\mu x$ intersects the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ in points whose abscissae are given by

$$x=\pm\frac{ab}{\sqrt{b^2-a^2\mu^2}}.$$

For these points to be real μ must be less than b/a. Thus if both the conjugate diameters $y=mx$ and $y=m_1x$ intersect the hyperbola in real points, m and m_1 both must be less than b/a. This contradicts the above condition of conjugacy. Hence, unlike the case of the ellipse, of any two conjugate diameters of a hyperbola, only one intersects it in real points.

7·52. If P be a point $(a \sec \phi, b \tan \phi)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the diameter PCP' is given by

$$y=\frac{b}{a}\sin \phi \, x.$$

If DCD' be the corresponding conjugate diameter its equation is by Art. 7·50

$$y=\frac{b}{a \sin \phi}x.$$

From what has been said above the points D and D' do not lie on the hyperbola. They have, however, a geometrical significance which is explained in the next article.

7·53. A hyperbola having the transverse and conjugate axes of a given hyperbola as its conjugate and transverse axes is called its **Conjugate Hyperbola.** Thus

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1 \quad or \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

is the equation to the hyperbola conjugate to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

The conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is intersected by the conjugate diameter $y=\frac{b}{a \sin \phi}x$ in points $(\pm a \tan \phi, \pm b \sec \phi)$. We designate these as the points D and D'. That this designation is not purely arbitrary will be borne out by the properties of conjugate diameters analogous to those for the ellipse. [Art. 7·55.]

It could easily be shown that the diameters conjugate with respect to a hyperbola are also conjugate with respect to its conjugate hyperbola, and that the property of conjugacy is reciprocal.

7·54. When the equation to a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

the equation to the asymptotes is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

and that to the conjugate hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

These three equations differ only in their constant terms. Also *the difference between the constant terms of the first and second is the same as that between the constant terms of the second and third*; and this will be seen to hold good in whatever manner these equations are transformed.

With the aid of this property the equation to a hyperbola being given, that to its conjugate can be at once written down. Thus the equation to the conjugate of the hyperbola

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

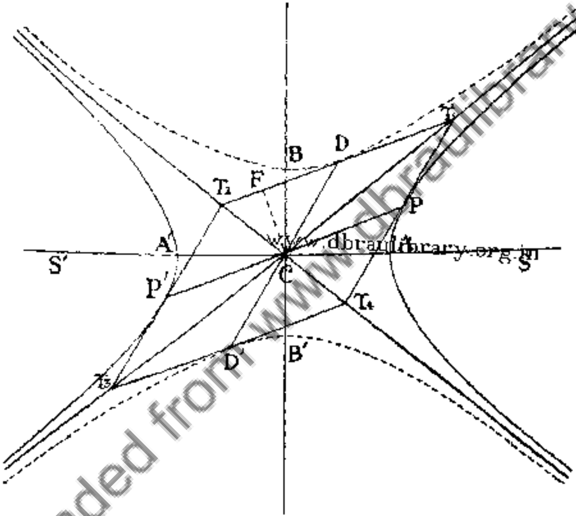
is $ax^2+2hxy+by^2+2gx+2fy+c=\dfrac{2\triangle}{ab-h^2}$, since the equation to the asymptotes was

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\triangle}{ab-h^2}. \qquad \text{Art. 7·48.}$$

7·55. Some properties of conjugate diameters :—

If PCP' and DCD' be two conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, P and D being the points

$$(a \sec \phi, b \tan \phi) \text{ and } (a \tan \phi, b \sec \phi)$$



respectively, then

(α) *Tangents at* **P** *and D are parallel to CD and CP.*

Tangent at P is

$$\frac{x \sec \phi}{a} - \frac{y \tan \phi}{b} = 1.$$

$$or \qquad y = \frac{b}{a \sin \phi} x - b \cot \phi,$$

a straight line parallel to CD.

Similarly tangent at D is

$$\frac{x \tan \phi}{a} - \frac{y \sec \phi}{b} = -1.$$

$$or \quad y = \frac{b}{a} \sin \phi \quad x + b \cos \phi,$$

a straight line parallel to CP.

(β) $\quad CP^2 - CD^2 = a^2 - b^2$

Now $\quad CP^2 = a^2 \sec^2 \phi + b^2 \tan^2 \phi$

and $\quad CD^2 = a^2 \tan^2 \phi + b^2 \sec^2 \phi$

Hence $CP^2 - CD^2 = a^2 - b^2$.

(γ) *The vertices of the parallelogram $T_1 T_2 T_3 T_4$ formed by the tangents at P, D, P' and D' lie on the asymptotes. Also area $T_1 T_2 T_3 T_4$ is $4ab$.*

Tangents at P and D are

$$\frac{x \sec \phi}{a} - \frac{y \tan \phi}{b} = 1$$

and $$\frac{x \tan \phi}{a} - \frac{y \sec \phi}{b} = -1$$

Adding $$\frac{x}{a} - \frac{y}{b} = 0$$

i.e., this point of intersection lies on the asymptote $\frac{x}{a} - \frac{y}{b} = 0$.

Similarly for the other points of intersection.

Also $T_1 T_2 T_3 T_4 = 4\, T_1 PCD$

$= 4\, CP.CF$

where F is the foot of the perpendicular from C on the tangent at D.

Now $\quad CP=\sqrt{a^2 \sec^2 \phi+b^2 \tan^2 \phi}$

$$=\sqrt{a^2+b^2 \sin^2 \phi}/\cos \phi$$

and $CF=1\bigg/\sqrt{\dfrac{\tan^2 \phi}{a^2}+\dfrac{\sec^2 \phi}{b^2}}$

$$=ab \cos \phi/\sqrt{a^2+b^2 \sin^2 \phi}$$

Hence $\quad T_1 T_2 T_3 T_4=4ab.$

(i) $\qquad SP.\ S'P=CD^2$

Now $\qquad SP=(ae \sec \phi-a)$

and $\quad S'P=(ae \sec \phi+a)$

$\therefore\ SP.\ S'P=a^2e^2 \sec^2 \phi-a^2$

$$=(a^2+b^2) \sec^2 \phi-a^2$$

$$=a^2 \tan^2 \phi+b^2 \sec^2 \phi$$

$$=CD^2.$$

Exercises.

1. Find the length of the semi-diameter conjugate to the diameter $y=3x$ of the hyperbola $9x^2-4y^2=36$.

2. Find the equation to the conjugate of the hyperbola

$$3x^2-5xy-2y^2+5x+11y-8=0$$

3. Shew that the asymptotes of

$$ax^2+2hxy+by^2=1$$

are conjugate diameters of $Ax^2+2Hxy+By^2=1$, if $aB+bA=2hH$.

4. Prove that the chords of a hyperbola, which touch the conjugate hyperbola, are bisected at their points of contact.

5. Shew that if two hyperbolas are conjugate to each other, the polar of a point on either of them with respect to the other is a tangent to the first.

6. If the polar of any point with respect to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ passes through one end of the conjugate axis, prove that the pole will lie on the tangent to the conjugate hyperbola at the other end of that axis.

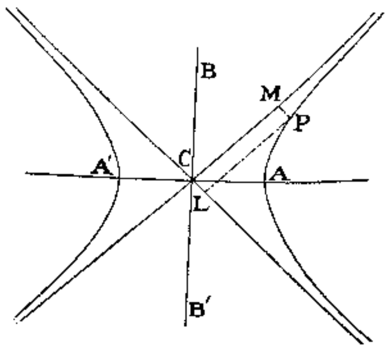
7. If the diameter PCP' meets the hyperbola at P and P', and the conjugate diameter meets the conjugate hyperbola at D and D', the asymptotes bisect PD and PD'.

8. If a chord be drawn through any point Q on a hyperbola to meet the asymptotes in R and R'; and CD, the diameter parallel to the chord, meets the conjugate hyperbola at D, shew that $QR. \ QR' = CD^2$.

9. Shew that the polar of any point on the conjugate hyperbola with respect to the original hyperbola touches the conjugate hyperbola.

7·60. *To find the equation to the hyperbola referred to its asymptotes as the axes of co-ordinates.*

Let P be a point on the hyperbola whose co-ordinates referred to CA and CB as axes are $(a \sec \phi, \ b \tan \phi)$. Let these referred to the asymptotes as axes be (X, Y).



Draw PL and PM parallel to the

asymptotes meeting them in L and M. Then $PL=X$ and $CM=Y$.

Now equations to PM and CM are

$$y=-\frac{b}{a}x+b\tan\phi+b\sec\phi$$

and $\quad y=\frac{b}{a}x.$

Whence the point of intersection M is given by

$$\left\{\frac{a(\tan\phi+\sec\phi)}{2},\ \frac{b(\tan\phi+\sec\phi)}{2}\right\}$$

Thus $X=CL=PM$

$$=\sqrt{\left\{a\,\frac{\sec\phi-\tan\phi}{2}\right\}^2+\left\{b\,\frac{\sec\phi-\tan\phi}{2}\right\}^2}$$

$$=\frac{\sec\phi-\tan\phi}{2}\sqrt{a^2+b^2}$$

and $\quad Y=CM=\dfrac{\tan\phi+\sec\phi}{2}\sqrt{a^2+b^2}$

$$\therefore\ XY=\frac{a^2+b^2}{4},\ \text{the required equation.}$$

Sometimes $\dfrac{a^2+b^2}{4}$ is replaced by c^2 and the above equation is written as

$$XY=c^2.$$

7·61. In terms of a single variable t the co-ordinates of any point on the hyperbola having the above equation may be taken as $\left\{ct,\ \dfrac{c}{t}\right\}$.

7·62. The tangent at any point 't' will be easily seen to be $X+Yt^2=2ct$.

This intersects the asymptotes (axes of X and Y) at $(2ct, 0)$ and $\left\{0, \frac{2c}{t}\right\}$. Hence follows a very important property of the hyperbola *viz*.

The part of the tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact.

The above result could also have been deduced from the property proved in Art. 7·55 (γ)

Ex. 1. If the tangent and normal to a rectangular hyperbola make intercepts a_1, a_2 on one asymptote and b_1, b_2 on the other, prove that

$$a_1a_2+b_1b_2=0.$$

Ex. 2. Prove that the rectangle contained by the intercepts made by any tangent to a hyperbola on its asymptotes is constant.

Ex. 3. A rectangular hyperbola is cut by a circle in four points. Prove that the sum of the squares of the distances of these four points from the centre of the hyperbola is equal to the square on the diameter of the circle.

Ex. 4. Shew that the points from which the sum of the squares of the normals to a rectangular hyperbola is constant lie on a circle.

Ex. 5. If one of the common chords of a circle and a rectangular hyperbola is a diameter of the hyperbola, shew that another of the common chords is a diameter of the circle.

Ex. 6. Shew that the two portions of a chord intercepted between a hyperbola and its asymptotes are equal.

Miscellaneous Exercises.

1. A straight line touches the circle which has for its diameter the line joining the foci of the hyperbola $b^2x^2-a^2y^2=a^2b^2$. Shew that the locus of its pole with respect to the hyperbola is an ellipse, whose eccentricity is

$$\frac{\sqrt{a^4-b^4}}{a^2}.$$

2. Find the equations of the common tangents of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

3. Prove that the locus of the poles of the chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which subtend a right angle at the fixed point (h, k) is

$$\frac{h^2+k^2+b^2}{a^2} x^2 - \frac{h^2+k^2-a^2}{b^2} y^2 - 2(hx+ky) + a^2 - b^2 = 0.$$

4. Prove that the radius of the circle, which touches the asymptotes and the hyperbola, is equal to the part of the latus-rectum produced intercepted between the curve and the asymptote.

5. Two tangents to a parabola are such that the external angle between them is constant and equal to α. Prove that the locus of their point of intersection is a hyperbola whose asymptotes are inclined at an angle 2α.

6. A variable point P lies on a fixed diameter $x=y$ of the ellipse $2x^2+3xy+4y^2=1$. Shew that the locus of the foot of the perpendicular from P upon its polar is the rectangular hyperbola $77(x^2-y^2)+72xy=8$.

7. A hyperbola has asymptotic directions parallel to the lines $x-2y=0$ and $3x-2y=0$, and passes through the origin where its tangent is $x-y=0$. Shew that its centre lies on the y- axis

8. Shew that the distance between the points of contact of a common tangent between two rectangular hyperbolas, the axes of one of which coincide with the

asymptotes of the other, is $\frac{1}{aa'}(a^4+a'^4)^{\frac{3}{4}}$, where $2a$ and $2a'$ are the transverse axes.

9. Shew that the circles whose diameters are chords of a rectangular hyperbola drawn parallel to a given direction constitute a co-axal system; and that the systems corresponding to two directions at right angles are orthogonal to one another.

10. Shew that the locus of the poles with respect to the parabola $y^2=4ax$ of tangents to the hyperbola $x^2-y^2=a^2$ is the ellipse $4x^2+y^2=4a^2$.

11. Chords of a hyperbola are drawn, all passing through the fixed point (h, k). Prove that the locus of their middle points is a hyperbola whose centre is the point $\left\{\frac{h}{2}, \frac{k}{2}\right\}$ and which is similar to either the hyperbola or its conjugate.

12. At any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tangent is drawn to meet the asymptotes in Q and R. If C be the centre of the hyperbola, shew that the locus of the centre of the circle circum-scribing the triangle CQR is

$$4(a^2x^2-b^2y^2)=(a^2+b^2)^2.$$

13. If the normals at two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meet on the line $lx+my=1$, prove that the tangent at these points meet on the curve

$$\frac{1}{a^2+b^2}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}=xy\left\{\frac{mx}{a^2}+\frac{ly}{b^2}\right\}+lx-my.$$

14. S is a focus of a conic, and PQ any chord subtending a right angle at S. Prove that the locus of the intersection of the tangents at P and Q is a conic whose focus is S. Find the latus-rectum and the eccentricity of this latter conic. [*Agra 1932*]

15. From any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ three normals are drawn other than the one at the point. Prove that the locus of the centroid of the triangle formed by their feet is $9\left\{\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}=\frac{a^2-b^2}{(a^2+b^2)^2}$, and that of the circum-centre is

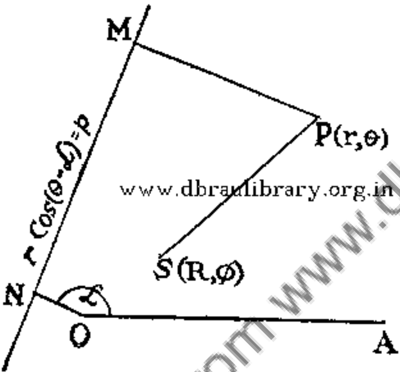
$$4(a^6x^2-b^6y^2)=a^4b^4.$$

16. If from a given point within an ellipse four normals are drawn to the ellipse, their points of intersection with the ellipse will lie upon a rectangular hyperbola, whose asymptotes are parallel to the axes and which passes through the given point.

CHAPTER VIII.

POLAR CO-ORDINATES.

8·10. *To find the general form of the polar equation to a conic.*



Let (R, ϕ) be the co-ordinates of the focus S, and let $r\cos(\theta-\alpha)=p$ be the equation to the directrix NM.

Take any point on the conic, and let its co-ordinates be (r, θ).

Now $SP^2=e^2PM^2$.

$$\therefore r^2+R^2-2rR\cos(\theta-\phi)=e^2\{r\cos(\theta-\alpha)-p^*\}^2 \quad \ldots(1)$$

is the required equation.

8·11. *To find the simplest form of the polar equation to a conic.*

If the focus S itself be taken as the pole then in (1) above

$$R=0, \ \phi=0$$

and $\quad p=SZ=\dfrac{SL}{e}=\dfrac{l}{e}$, [*Fig. next page*]

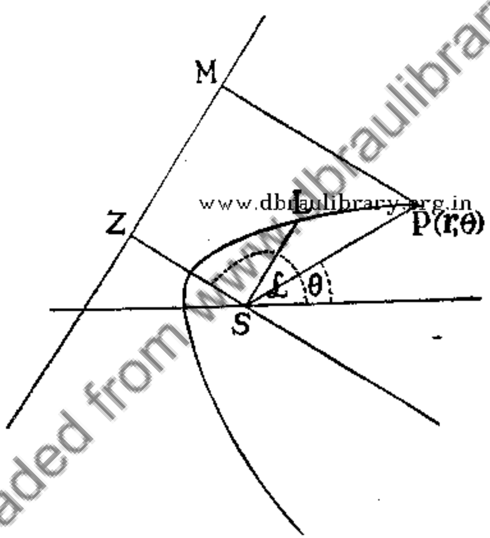
*That the perpendicular from (r, θ) upon $r\cos(\theta-\alpha)=p$ is $r\cos(\theta-\alpha)-p$ is easily seen by a reference to the Cartesian form of the equation.

Hence the equation to the conic reduces to

$$r = \pm e \left\{ r \cos(\theta - \alpha) - \frac{l}{e} \right\}$$

i.e. either $\dfrac{l}{r} = -1 + e \cos(\theta - \alpha)$ ....(2)

or $\dfrac{l}{r} = 1 + e \cos(\theta - \alpha)$ .... (3)



where α is the angle which the perpendicular from the focus upon the directrix makes with the initial line.

Either of the equations (2) or (3) may be taken as the equation to the conic under the given conditions. It is the latter form that is generally used.

8·12. Obviously the above equation could be further simplified by considering ZS itself as the initial line.

In this case $\alpha=\pi$, and the equation to the conic reduces to

$$\frac{l}{r}=1-e\cos\theta \qquad \text{.......................(4)}$$

If the positive direction of the initial line be SZ and not ZS the corresponding equation to the conic will be found by putting $\alpha=0$ in (3). It is

$$\frac{l}{r}=1+e\cos\theta \qquad \text{.......................(5)}$$

Ex. 1. If SP, a focal radius of a conic, is produced to Q, so that $SQ=k.\ SP$, where k is constant, prove that the locus of Q is a conic of equal eccentricity with the given conic, and latus rectum k times that of the given conic.

Ex. 2. Find the equation of the directrix of the conic $\frac{l}{r}=1+e\cos(\theta-\alpha)$ corresponding to the origin. [*Agra* 1929]

Ex. 3. The latus rectum of a conic is 5 and its eccentricity 2/5. Find the length of the focal chord making an angle 60° with the major axis.

Ex. 4. Shew that the equations $\frac{l}{r}=1-e\cos\theta$, and $\frac{l}{r}=-e\cos\theta-1$ represent the same conic.

8·20. To find the shape of the conic given by any one of the equations of the previous article, we shall be required to find a sufficient number of points on it, by giving values to θ successively from 0 to 2π, and determining the corresponding values for r. For $e=1$ or <1 *i.e.*, for the parabola or the ellipse this will present no difficulty and is left as an exercise to the reader. But the case of the hyperbola ($e>1$) deserves careful attention.

Let the equation to the hyperbola be $\frac{l}{r}=1-e\cos\theta$.

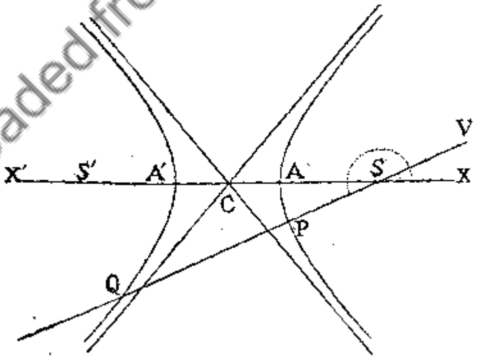
The following points may be noted :—

(i) For $\theta=0$, r is negative and equal to $\dfrac{l}{1-e}$ or $-a(1+e)$. The corresponding point is A'.

(ii) As θ increases from 0 to $\cos^{-1} 1/e$, r steadily decreases from $-a(1+e)$ to $-\infty$. (r, θ) traces out the lower half of the left hand branch of the curve.

(iii) As θ passes through $\cos^{-1} 1/e$, r changes from $-\infty$ to ∞, and from an infinite distance on the lower half of the left hand branch, (r, θ) changes over to an infinite distance on the upper half of the right hand branch.

(iv) As θ increases from $\cos^{-1} 1/e$ to π, r decreases from ∞ to $a(1-e)$, (r, θ) tracing out the upper half of the right hand branch.



(v) For $\theta=\pi$, $r=a(1-e)$. The corresponding point is A.

(*vi*) As θ increases from π to $2\pi - \cos^{-1} 1/e$, r increases from $a(1+e)$ to ∞, and (r, θ) traces out the lower half of the right hand branch.

(*vii*) As θ passes through $2\pi - \cos^{-1} 1/e$, r changes from ∞ to $-\infty$, and from an infinite distance on the lower half of the right hand branch, (r, θ) changes over to an infinite distance on the upper half of the left hand branch,

and (*viii*) As θ increase from $2\pi - \cos^{-1} 1/e$ to 2π, r increases from $-\infty$ to $-a(1+e)$, and (r, θ) traces out the upper half of the left hand branch, completing the tracing of the hyperbola.

8·21. From what has been said in (*ii*) and (*viii*) above, it should be clear that the radius vector of any point on the left hand branch of the hyperbola, given by $\frac{l}{r} = 1 - e \cos \theta$, is always negative; and that of any point on the right hand branch is always positive. This should be carefully borne in mind while finding out the corresponding vectorial angles *e.g.* in the figure of the last article the vectorial angle of the point Q is XSV, whereas that of the point P is XSP.

8·30. *To find the equation to the chord joining the points* (r_1, θ_1) *and* (r_2, θ_2) *on the conic*

$$\frac{l}{r} = 1 - e \cos \theta.$$

Also to find the equation to the tangent to the conic at the point (r_1, θ_1).

The equation to a straight line passing through any two points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{\sin(\theta_1 - \theta_2)}{r} = \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2} \quad \ldots(1) \quad \text{[Art. 2·13]}$$

Since these points lie on the given conic

$$\frac{l}{r_1}=1-e\cos\theta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{l}{r_2}=1-e\cos\theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Substituting for r_1 and r_2 from (2) and (3) in (1),

$$\frac{l}{r}\sin(\theta_1-\theta_2)=\sin(\theta-\theta_2)(1-e\cos\theta_1)$$
$$+\sin(\theta_1-\theta)(1-e\cos\theta_2)$$

$$=\{\sin(\theta-\theta_2)+\sin(\theta_1-\theta)\}$$
$$-e\{\sin(\theta-\theta_2)\cos\theta_1+\sin(\theta_1-\theta)\cos\theta_2\}$$

$$=2\sin\frac{\theta_1-\theta_2}{2}\cos\left\{\theta-\frac{\theta_1+\theta_2}{2}\right\}$$

$$-\frac{e}{2}\{\sin(\theta+\theta_1-\theta_2)+\sin(\theta-\theta_1-\theta_2)$$
$$+\sin(\theta_1-\theta+\theta_2)+\sin(\theta_1-\theta-\theta_2)\}$$

$$=2\sin\frac{\theta_1-\theta_2}{2}\cos\left\{\theta-\frac{\theta_1+\theta_2}{2}\right\}$$
$$-e\sin(\theta_1-\theta_2)\cos\theta$$

or $$\frac{l}{r}=\sec\frac{\theta_1-\theta_2}{2}\cos\left\{\theta-\frac{\theta_1+\theta_2}{2}\right\}-e\cos\theta,$$

is the required equation to the chord.

8·31. Substituting $\theta_2=\theta_1$ in the above, we get

$$\frac{l}{r}=\cos(\theta-\theta_1)-e\cos\theta$$

for the equation to the tangent.

Ex. 1. If a chord subtends a constant angle 2α at a focus, shew that the locus of the point where it meets the internal bisector of that angle is

$$\frac{l}{r}=\sec\alpha-e\cos\theta.$$ [*Agra 1938*]

Ex. 2. Find the angle between the radius vector and the tangent at any point of the conic

$$\frac{l}{r}=1-e\cos\theta.$$

Ex. 3. A system of conics have the same focus and latus rectum. Prove that the tangents at all points on a fixed line through the focus cut the latus rectum produced at the same distance from the focus.

Ex. 4. Find the polar co-ordinates of the point of intersection of the tangents at the points, whose vectorial angles are α and β; and hence prove that if the tangents to a parabola at P and Q meet in T, then $ST^2=SP.\,SQ$.

Ex. 5. Shew that the portion of a tangent intercepted between the directrix and the point of contact subtends a right angle at the focus.

8·4. *To find the equation to the normal to the conic* $\frac{l}{r}=1-e\cos\theta$ *at the point* (r_1, θ_1) *on it.*

The equation to the tangent at the point is

$$\frac{l}{r}=\cos(\theta-\theta_1)-e\cos\theta.$$

$$\text{or}\quad r\cos(\theta-\beta)=l,$$

$$\left.\begin{aligned}\text{where}\quad \cos\beta&=\cos\theta_1-e\\ \text{and}\quad \sin\beta&=\sin\theta_1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(2)$$

Any straight line perp. to (1) is given by

$$r\sin(\theta-\beta)=A.$$

If it passes through (r_1, θ_1) its equation is

$$r\sin(\theta-\beta)=r_1\sin(\theta_1-\beta)$$

$$\text{or}\quad r\{\sin\theta\cos\beta-\cos\theta\sin\beta\}$$

$$=r_1\{\sin\theta_1\cos\beta-\cos\theta_1\sin\beta\}$$

$$\text{or}\quad r\{\sin(\theta-\theta_1)-e\sin\theta\}$$

$$=\ -e\,r_1\sin\theta_1$$

$$\text{or} \quad \frac{l}{r} \quad \frac{e \sin \theta_1}{1-e\cos\theta_1} = e \sin \theta - \sin (\theta - \theta_1)$$

$$\left\{ \because \frac{l}{r_1} = 1 - e \cos \theta_1 \right\}$$

the required equation to the normal.

8·5. *To find the equation to the chord of contact of tangents* drawn from a point* (r_1, θ_1) *to the conic* $\frac{l}{r} = 1 - e \cos \theta$.

Let the vectorial angles of the two points of contact be α and β.

Then the equation to the chord through them is

$$\frac{l}{r} = \sec \frac{\alpha - \beta}{2} \cos \left\{ \theta - \frac{\alpha + \beta}{2} \right\} - e \cos \theta \text{ ....(1)}$$

Also

$$\frac{l}{r_1} = \cos (\theta_1 - \alpha) - e \cos \theta_1 \text{ ....(2)}$$

and $$\frac{l}{r_1} = \cos (\theta_1 - \beta) - e \cos \theta_1 \text{ ....(3)}$$

Subtracting (3) from (2)

$$\cos (\theta_1 - \alpha) = \cos (\theta_1 - \beta)$$

or $$\theta_1 - \alpha = -(\theta_1 - \beta) \qquad [\because \alpha \neq \beta].$$

or $$\frac{\alpha + \beta}{2} = \theta_1 \text{ ....(4)}$$

Substituting from (4) in (3)

$$\cos \frac{\alpha - \beta}{2} = \frac{l}{r_1} + e \cos \theta_1 \text{ ....(5)}$$

With the help of (4) and (5), equation (1) becomes

$$\left\{ \frac{l}{r} + e \cos \theta \right\} \left\{ \frac{l}{r_1} + e \cos \theta_1 \right\} = \cos \left\{ \theta - \theta_1 \right\},$$

the required equation,

*The equation to these tangents themselves does not come out in a handy form. So we do not propose to discuss it.

It may be observed that this is also the equation to the polar of the point (r_1, θ_1).

Ex. 1. If the tangents at any two points P and Q meet in T, and if PQ meets the corresponding directrix in K, shew that KST is a right angle.

Ex. 2. Find the locus of the pole of a chord which subtends a constant angle 2α at a focus of a conic distinguishing the cases for which $\cos\alpha > =$ and $< e$. [*Agra 1934*]

8·6. *To find the equation to the Director Circle of the conic* $\dfrac{l}{r}=1-e\cos\theta$.

The equation to the tangent at any point θ_1 is

$$\frac{l}{r}=\cos(\theta-\theta_1)-e\cos\theta=\cos(\theta-\alpha) \quad \ldots\ldots\ldots\ldots(1)$$

where $\cos\alpha=\cos\theta_1-e$ and $\sin\alpha=\sin\theta_1$

Similarly the tangent at θ_2 is given by

$$\frac{l}{r}=\cos(\theta-\theta_2)-e\cos\theta=\cos(\theta-\beta) \quad \ldots\ldots\ldots\ldots(2)$$

where $\cos\beta=\cos\theta_2-e$ and $\sin\beta=\sin\theta_2$.

If these intersect at (R, ϕ), we have

$$\frac{l}{R}=\cos(\phi-\theta_1)-e\cos\phi \quad \ldots\ldots\ldots\ldots(3)$$

and

$$\frac{l}{R}=\cos(\phi-\theta_2)-e\cos\phi \quad \ldots\ldots\ldots\ldots(4)$$

$$\therefore \cos(\phi-\theta_1)=\cos(\phi-\theta_2)$$

$$\text{or} \quad \phi-\theta_1=-(\phi-\theta_2)$$

$$\text{or} \quad \phi=\frac{\theta_1+\theta_2}{2} \quad \ldots\ldots\ldots\ldots(5)$$

Substituting this value of ϕ in (4), we get

$$\cos\frac{\theta_1-\theta_2}{2}=\frac{l}{R}+e\cos\phi \quad \ldots\ldots\ldots\ldots(6)$$

Also since (1) and (2) are at right angles

$$\alpha-\beta=\frac{\pi}{2},$$

i.e. $1+\tan\alpha\tan\beta=0$

i.e. $(\cos\theta_1-e)(\cos\theta_2-e)+\sin\theta_1\sin\theta_2=0$

or $e^2-e(\cos\theta_1+\cos\theta_2)+\cos(\theta_1-\theta_2)=0$

or $e^2-2e\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}+2\cos^2\frac{\theta_1-\theta_2}{2}-1=0$

i.e. $e^2-2e\cos\phi\left\{\frac{l}{R}+e\cos\phi\right\}+2\left\{\frac{l}{R}+e\cos\phi\right\}^2-1=0$

or $\frac{2l^2}{R^2}+\frac{2el}{R}\cos\phi-1+e^2=0$

Generalising the co-ordinates, we obtain the required equation

$$r^2(e^2-1)+2erl\cos\theta+2l^2=0.$$

Miscellaneous Exercises.

1. Prove that two equal conics which have a common focus and whose axes are inclined at an angle 2α intersect at an angle

$$\tan^{-1}\left\{\frac{e^2\sin2\alpha+2e\sin\alpha}{e^2\cos2\alpha+2e\cos\alpha+1}\right\}.$$

2. Shew that the polar equation of the auxiliary circle of the conic $\frac{l}{r}=1+e\cos\theta$ is

$$r^2(1-e^2)-2ler\cos\theta+l^2=0.$$

3. If, with the focus of a parabola as centre, a circle be described passing through the vertex, the rectangle under the intercepts of any focal chord between the circle and the parabola is constant.

4. Two conics have the same focus and directrix. If any tangent to one cuts the other in P and Q, shew that $\angle PSQ$ is constant, and $\cos \frac{1}{2} PSQ = \frac{e}{e'}$ where S is the common focus and e and e' are the eccentricities.

5. If a chord PQ of the conic $\frac{l}{r} = 1 + e \cos \theta$ subtends a constant angle β at the focus, shew that PQ touches a conic having the same focus and directrix as the given conic.

6. If a focal chord of an ellipse makes an angle α with the axis, shew that the angle between the tangents at its extremities is $\tan^{-1} 2e \sin \alpha/(1-e^2)$.

7. Prove that the sum of the reciprocals of two perpendicular focal chords of an ellipse is constant, and that its semi-latus rectum is the harmonic mean between the segments of any focal chord. [*Benares 1932*]

8. Prove that the locus of the intersection of the tangents at the extremities of the perpendicular focal radii of a conic is another conic having the same focus.

9. If the tangent at any point P of a conic meets the directrix in K, shew that the angle KSP is a right angle. [*Agra 1928*]

10. A circle passes through the focus of a conic, and meets it in four points, shew that

(*i*) the continued product of the distances of the points of intersection from the focus is constant,

and (*ii*) the sum of the reciprocals of these distances is constant.

11. Two equal ellipses of eccentricity e, are placed with their axes at right angles and they have one focus S in common. If PQ be a common tangent, shew that the angle PSQ is equal to $2 \sin^{-1} e/\sqrt{2}$. [*Benares 1933*]

CHAPTER IX.

TRACING OF CONICS.

9·1. We have seen that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent any of the five conics *viz.*,

(1) A pair of straight lines $(\Delta=0)$.

(2) A circle $(\Delta \neq 0, a=b, h=0)$.

(3) A parabola $(\Delta \neq 0, h^2=ab)$.

(4) An ellipse $(\Delta \neq 0, h^2<ab)$.

(5) A hyperbola $(\Delta \neq 0, h^2>ab)$.

Any equation of the second degree in x and y being given, the category, to which the conic given by it belongs, can be easily ascertained. If it belongs to either of the first two categories, its tracing is a simple affair. Here we shall concern ourselves with the tracing of the last three. To trace these the following data will be required :—

For Central Conics i.e., ellipses or hyperbolas :—

Centre, directions and lengths of axes, foci.

For parabolas :—Axis, vertex and focus.

9·20. *To find the co-ordinates of the centre of the Central conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
$$(\Delta \neq 0, h^2 \neq ab).$$

One method for finding out these co-ordinates is indicated in Art. 4·51. Here is another.

Let the co-ordinates of the centre be (x_1, y_1). Transferring the origin to the centre, the transformed equation of the conic becomes

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2$$
$$+2g(x+x_1)+2f(y+y_1)+c=0.$$

or $$ax^2+2hxy+by^2$$
$$+2x(ax_1+hy_1+g)+2y(hx_1+by_1+f)$$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \ldots(1)$$

Now since the origin is the centre of the conic given by (1), corresponding to every point (x, y) on it, there must also be a point $(-x, -y)$ on it. This is possible only if the co-efficients of x and y are zeros simultaneously,

i.e., if $$\left.\begin{array}{l} ax_1+hy_1+g=0 \\ hx_1+by_1+f=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Equations (2) give the co-ordinates of the centre *viz.*,

$$x_1=\frac{hf-bg}{ab-h^2},\ y_1=\frac{gh-af}{ab-h^2}$$

Cor. 1. *The absolute term in the transformed equation is* $\dfrac{\Delta}{ab-h^2}$, *where* $\Delta \equiv abc+2fgh-af^2-bg^2-ch^2$.

Cor. 2. *The equation to the curve referred to the centre as origin can be written down by substituting half the co-ordinates of the centre for x and y in the first degree terms of the original equation.*

Ex. **1.** Find the centre of the conic

$$2x^2-5xy-3y^2-x-4y+6=0.$$

Ex. **2.** Find the equation to the conic

$$7x^2-9xy+4y^2-32x+25y+47=0.$$

referred to its centre as origin.

Ex. 3. Find the absolute term of the transformed equation when the origin is transferred to the centre of the conic

$$7x^2-12xy-2y^2+2x+4y+2=0.$$

9·21. *To find the directions of the axes of the Central conic* $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Referred to its centre as origin, the above equation becomes

$$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now in order that the axes of the conic may coincide with the axes of co-ordinates, the latter should be so rotated that the equation into which (1) transforms may have no terms containing xy. If θ be this angle of rotation

$$\tan 2\theta=\frac{2h}{a-b} \quad \text{(Art. 4·25 Lemma)} \quad \ldots\ldots(2)$$

The above is obviously a quadratic in $\tan\theta$, leading to two values for θ, which give the directions of the two axes of the conic.

Cor. 1. *The equation to the axes of the conic*

$$ax^2+2hxy+by^2=k$$

is
$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

i.e., the same as the equation to the bisectors of the angles between the asymptotes (real in the case of the hyperbola and imaginary in the case of ellipse) of the conic.

This is at once clear by changing (2) into cartesians.

Cor. 2. *The equation to the axes of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is
$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h}$$

The directions of the asymptotes are the same as in the case of Cor. 1. Only the centre is different.

Ex. 1. Find the directions of the axes of the conics :—

(*i*) $5x^2+3xy+y^2-7x+8y=0$

(*ii*) $6x^2-4xy+9y^2-24x-22y+43=0$

Ex. 2. Find the equations of the axes of the conic

(*i*) $x^2-xy+y^2+4x-5y-2=0.$

(*ii*) $5x^2-5y^2-24xy+14x+8y-16=0.$

9·22. *To find the lengths of the axes of the Central conic* $ax^2+2hxy+by^2+2gx+2fy+c=0.$

*First Method** :—

Changing equation (1) of Art. 9·21 into polars

$$\frac{k}{r^2}=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta \quad \ldots\ldots\ldots(1)$$

where k stands for $-\dfrac{\Delta}{ab-h^2}$

Each one of the values of θ obtained from equation (2) of the last article, when substituted in the above equation leads to a value for r^2, which gives the square of the corresponding semi-axis. In the case of hyperbola one of these will come out to be negative.

*Second Method** :—

Equation (1) of the last article when referred to axes of the conic as the axes of co-ordinates takes the form $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}=1$, where α, β are the lengths of the semi-axes.

*First Method may be applied when lengths corresponding to the directions of the axes are required, whereas the Second Method can be conveniently used if only the lengths are required.

From Invariants (Art. 4·25)

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=(a+b)\Big/-\frac{\Delta}{ab-h^2}.$$

and $$\frac{1}{\alpha^2\beta^2}=(ab-h^2)\Big/\frac{\Delta^2}{(ab-h^2)^2}\ \textit{i.e.}\ (ab-h^2)^3\Big/\Delta^2.$$

Now $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$ are evidently the roots of the equation

$$t^2-t\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+\frac{1}{\alpha^2\beta^2}=0.$$

or $$t^2+\frac{t\,(a+b)(ab-h^2)}{\Delta}+\frac{(ab-h^2)^3}{\Delta^2}=0.$$

Ex. 1. Find the lengths of the axes of the conic

$$x^2+xy+y^2=8.$$

Ex. 2. Shew that the semi-axes of the conic

$$ax^2+2hxy+ay^2=c$$

are given by $\sqrt{\dfrac{c}{a-h}}$ and $\sqrt{\dfrac{c}{a+h}}$.

Ex. 3. Shew that the area of the ellipse

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is equal to $\pi\dfrac{\Delta}{(ab-h^2)^{\frac{3}{2}}}$.

9·23. *To find the foci of the Central conic*

$$ax^2+2hxy+by^2=1.$$

First Method :—

If θ be the angle at which the major axis is inclined to the axis of x, the equation to the major axis can be written as

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r.$$

The foci being on the major axis at distances $\pm\alpha e$ from the centre of the conic, where 2α is the length of its major axis and e its eccentricity, their co-ordinates will obviously be given by

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=\pm\alpha e$$

$$\text{or}\quad \left.\begin{aligned} x &= \pm\alpha e\cos\theta \\ y &= \pm\alpha e\sin\theta \end{aligned}\right\}$$

Second Method :

Lemma. *The equation to the pair of tangents to a conic from a focus is that of a point circle at the focus.*

Let the focus be taken as the origin of co-ordinates, and the corresponding directrix as a straight line parallel to the axis of y, say $x=h$. Then the equation to the conic is

$$x^2+y^2=e^2(x-h)^2$$

$$\text{or}\quad x^2(1-e^2)+2e^2hx+y^2=e^2h^2.$$

Equation to the pair of tangents (imaginary of course) from the focus to the conic is (Art. 4·41)

$$[x^2(1-e^2)+2e^2hx+y^2-e^2h^2](-e^2h^2)=[e^2hx-e^2h^2]^2$$

$$\text{or}\quad x^2+y^2=0. \qquad \text{[on simplification]}$$

Let (x', y') be the focus of the given conic. The equation to the pair of tangents from it to the conic is

$$(ax^2+2hxy+by^2-1)(ax'^2+2hx'y'+by'^2-1)$$
$$=\{axx'+h(xy'+x'y)+byy'-1\}^2$$

In view of the above Lemma this equation must represent a circle, hence

$$a(ax'^2+2hx'y'+by'^2-1)-(ax'+hy')^2$$
$$=b(ax'^2+2hx'y'+by'^2-1)-(hx'+by')^2\ldots\ldots \text{(i)}$$

$$\text{and}\quad 2h(ax'^2+2hx'y'+by'^2-1)$$
$$-2(ax'+hy')(hx'+by')=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$i.e. \quad \frac{x'^2 - y'^2}{a-b} = \frac{1}{h^2 - ab}$$

$$\text{and} \qquad \frac{x'y'}{h} = \frac{1}{h^2 - ab}$$

The foci, therefore, are obtained by solving the equations

$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h} = \frac{1}{h^2 - ab} \quad \text{..................}(iii)$$

9·24. *To find the foci of the central conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If (x', y') be the co-ordinates of the centre of the conic, the given equation referred to it as the origin can be written down (Cor. 1 Art. 9·20) as

$$ax^2 + 2hxy + by^2 = \frac{\triangle}{h^2 - ab} = k \text{ (Say)} \quad \text{.........}(iv)$$

First Method :—

As in the previous article the co-ordinates of the foci will be easily seen to be given by

$$\frac{x - x'}{\cos\theta} = \frac{y - y'}{\sin\theta} = \pm\alpha' e'$$

where θ is the inclination to the axis of x of the major axis, $2\alpha'$ its length and e' the eccentricity of the given conic.

Second Method :—

Referred to the centre as the origin, the foci of the given conic will be given by

$$\frac{x^2 - y^2}{(a-b)/k} = \frac{xy}{h/k} = \frac{1}{(h^2 - ab)/k^2} \qquad \text{(Art. 9·23)}$$

$$i.e. \quad \frac{x^2 - y^2}{a-b} = \frac{xy}{h} = \frac{\triangle}{(h^2 - ab)^2} \quad \text{..................}(v)$$

The co-ordinates of the foci, referred to the original axes, therefore, are $(x+x', y+y')$ where (x', y') are those of the centre of the conic and x, y are the values obtained from equation (v).

Ex. **Find the foci of the following curves :—**

(*i*) $x^2+8xy-5y^2-2x+6y-6=0$.

(*ii*) $3x^2-4xy+2x+4y-9=0$.

(*iii*) $x^2-6xy+y^2-10x-10y-19=0$.

9·30. *To find the axis and the vertex of the parabola*

$$(\alpha x+\beta y)^2+2gx+2fy+c=0^* \quad \ldots\ldots\ldots\ldots(1)$$

Any straight line $\alpha x+\beta y+k=0$, where k is an arbitrary constant is parallel to the axis of the parabola†. Let the equation to the axis be $\alpha x+\beta y+\lambda=0$, where λ is to be determined.

Equation (1) can be written as

$$(\alpha x+\beta y+\lambda)^2=2x(\alpha\lambda-g)+2y(\beta\lambda-f)+\lambda^2-c \quad \ldots\ldots(2)$$

Let λ be so chosen that the straight lines

$$\alpha x+\beta y+\lambda=0 \quad \ldots\ldots\ldots\ldots(3)$$

$$\text{and} \quad 2x(\alpha\lambda-g)+2y(\beta\lambda-f)+\lambda^2-c=0 \quad \ldots\ldots(4)$$

are at right angles.

---

*Since $ab=h^2$, the general equation of the second degree to a parabola takes this form.

†All such straight lines will be seen to intersect parabola in one point at infinity. That every straight line intersecting the parabola in one point at infinity is parallel to the axis will be easily seen thus :—

For the points of intersection of the parabola $y^2=4ax$ and the straight line $y=mx+c$,

$$my^2-4ay+4ac=0.$$

If one of these points is at infinity, m must be zero. The straight line then reduces to $y=c$, a straight line parallel to the axis of the parabola.

For this $\frac{\alpha}{\beta} \cdot \frac{\alpha\lambda-g}{\beta\lambda-f}=-1$

$$\text{or} \quad \lambda=\frac{(\alpha g+\beta f)}{\alpha^2+\beta^2} \quad \text{.........................(5)}$$

With this value for λ, equation (2) takes the form

$$(\alpha x+\beta y+\lambda)^2=\frac{2(\alpha f-\beta g)}{\alpha^2+\beta^2}\left\{\beta x-\alpha y+k\right\} \quad \text{......(6)}$$

where k stands for $\dfrac{(\lambda^2-c)(\alpha^2+\beta^2)}{2(\alpha f-\beta g)}$

Referring to the standard form of its equation, a parabola could be defined as the locus of a point the square of whose distance from a given straight line (its axis) varies as its distance from a perpendicular straight line (the tangent at its vertex).

In this form equation (6) above can be written as

$$\left(\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}\right)^2=\pm\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{\frac{3}{2}}}\left(\frac{\beta x-\alpha y+k}{\pm\sqrt{\alpha^2+\beta^2}}\right) \quad \text{......(7)}$$

From (7) the axis, the tangent at the vertex and the latus-rectum of the parabola are seen to be

$$\alpha x+\beta y+\lambda=0$$

$$\beta x-\alpha y+k=0$$

and $\pm 2(\alpha f-\beta g)\Big/(\alpha^2+\beta^2)^{\frac{3}{2}}$ respectively.

The way to resolve the ambiguity in (7) above is shown in Solved Ex. 3 Art. 9·4.

9·31. *To find the focus of the parabola*

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \quad (h^2=ab).$$

If (x_1, y_1) be the vertex of the parabola and θ be the angle at which its axis is inclined to the x- axis, the co-ordinates x and y of the focus are obviously given by

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\frac{l}{2},$$

where l is the semi latus-rectum of the parabola.

Ex. 1. Find the equation to the axis, the co-ordinates of the vertex and the length of the latus rectum in each of the following parabolas :—

(*i*) $x^2-2xy+y^2+4x-8y+9=0$.

(*ii*) $4x^2+4xy+y^2+7x+16y-11=0$.

Ex. 2. Find the focus of the parabola

$$16x^2-24xy+9y^2-60x-80y=0.$$

Ex. 3. *Find the equation to the directrix of the parabola $x^2+2xy+y^2-3x+6y-4=0$; also the co-ordinates of the focus.

9·4. *Ex.* 1. *Trace the curve whose equation is*

$$11x^2+4xy+14y^2-26x-32y+23=0 \quad\ldots\ldots\ldots\ldots(1)$$

Since Δ in this case is equal to -900 and $ab-h^2=150$, the curve is an ellipse.

Solving the equations

$$11x+2y-13=0$$

and $$2x+14y-16=0$$

the centre comes out to be (1, 1).

Transferring the origin to this centre, (Cor. 2, Art. 9·20) the given equation becomes

$$11x^2+4xy+14y^2-13-16+23=0$$

or $$11x^2+4xy+14y^2=6 \quad\ldots\ldots\ldots\ldots(2)$$

The directions of the axes are given by

$$\tan 2\theta = -4/3$$

i.e. $$2\tan^2\theta-3\tan\theta-2=0$$

or $$\tan\theta=2, \quad or \quad -1/2.$$

*See Art. 4·43.

Changing (2) to polars

$$\frac{6}{r^2}=\frac{11+4\ tan\ \theta+14\ tan^2\ \theta}{1+tan^2\ \theta}$$

$$\therefore\quad \frac{6}{r_1^2}=\frac{11+8+56}{1+4}=15 \quad \text{when} \quad tan\ \theta=2$$

$$\text{and} \quad \frac{6}{r_2^2}=\frac{11-2+7/2}{1+1/4}=10 \quad \text{when} \quad tan\ \theta=-1/2.$$

Hence the standard form of the equation is

$$\frac{x^2}{2/5}+\frac{y^2}{3/5}=1.$$

Thus the given equation represents an ellipse, whose centre is (1, 1), whose major axis is inclined at an angle $tan^{-1}(-1/2)$ to the axis of x, and whose semi-axes are $\sqrt{3/5}$ and $\sqrt{2/5}$.

Ex. 2. Trace the conic represented by

$$6x^2+5xy-6y^2-4x+7y+11=0. \quad \text{..................}(1)$$

Since Δ in this case is equal to $-\frac{2197}{4}$ and $ab<h^2$, the equation represents a hyperbola.

Solving the equations

$$12x+5y-4=0$$
$$\text{and} \quad 5x-12y+7=0.$$

for the centre, we get $\left\{\frac{1}{13}, \frac{8}{13}\right\}$

On transferring the origin to the centre, the given equation becomes

$$6x^2+5xy-6y^2-4\cdot\frac{1}{26}+7\cdot\frac{4}{13}+11=0 \quad \text{...................}(2)$$

$$\text{or} \quad 6x^2+5xy-6y^2+13=0$$

The directions of the axes are given by

$$tan\ 2\theta=\frac{5}{12}$$

i.e. $\quad 5\ tan^2\ \theta+24\ tan\ \theta-5=0$

i.e. $\quad tan\ \theta=\frac{1}{5} \quad \text{or} \quad -5$

Changing (2) to polars

$$\frac{13}{r^2}=\frac{6\,tan^2\,\theta-5\,tan\,\theta-6}{1+tan^2\,\theta}$$

$$\frac{13}{r_1^{\,2}}=-\frac{13}{2}\qquad\text{when}\qquad tan\,\theta=\frac{1}{5}$$

and $$\frac{13}{r_2^{\,2}}=\frac{13}{2}\qquad\text{when}\qquad tan\,\theta=-5.$$

Hence the standard form of the equation is

$$\frac{x^2}{-2}+\frac{y^2}{2}=1.$$

Thus the given equation represents a rectangular hyperbola, whose centre is (1/13, 8/13) and whose semi-axes are each equal to $\sqrt{2}$, and whose transverse axis is inclined at an angle $tan^{-1}\ (-5)$ to the axis of x.

Ex. 3. *Trace the curve*

$$16x^2-24xy+9y^2-104x-172y+44=0 \qquad\ldots\ldots\ldots\ldots(1)$$

Since in this case $\Delta\neq 0$ and $ab=h^2$, the curve is a parabola.

The given equation can also be written as

$$(4x-3y+\lambda)^2=2x(4\lambda+52)+2y(86-3\lambda)+\lambda^2-44.$$

If $$4x-3y+\lambda=0$$

and $$2x(4\lambda+52)+2y(86-3\lambda)+\lambda^2-44=0$$

be at right angles, then

$$\frac{4}{3}\cdot\frac{4\lambda+52}{3\lambda-86}=-1,$$

$$i.e.\qquad \lambda=2.$$

The given equation takes the form

$$(4x-3y+2)^2=120x+160y-40.$$

$$=40(3x+4y-1)\qquad\ldots\ldots\ldots\ldots(2)$$

$$i.e.\quad\left(\frac{4x-3y+2}{5}\right)^2=\pm 8\ \frac{3x+4y-1}{\pm 5}.\qquad\ldots\ldots\ldots\ldots(3)$$

In order to ascertain the correct sign to be prefixed to the above expression for the perpendicular, we should find on which side of the tangent at the vertex the parabola lies. This will be decided with the help of its intercepts on the axes of x and y.

On the axis of x, these intercepts are given by

$$16x^2-104x+44=0,$$

which shows that they are imaginary. On the axis of y the equation

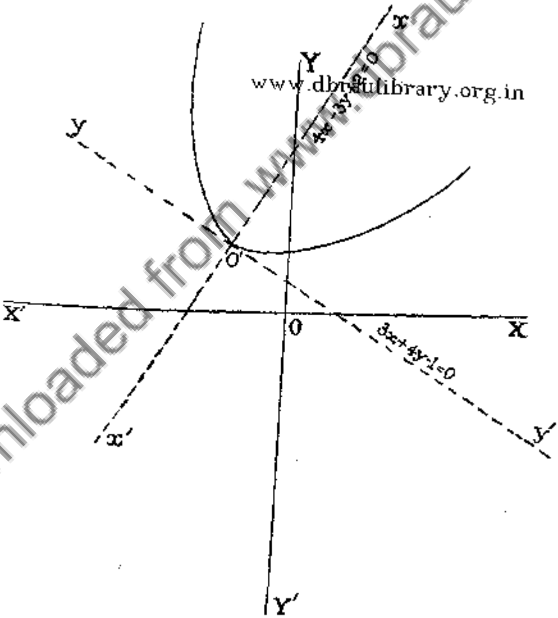
$$9y^2-172y+44=0$$

shews intercepts to be positive.

Relating these to the figure, it can at once be observed that the parabola lies to the right of $yo'y'$, the tangent at the vertex. Hence the expression for the perpendicular from any point on the parabola to the line $3x+4y-1=0$ should be preceded by the positive sign (Art. 2·2).

With proper sign, equation (3) can, therefore, be written as

$$\left(\frac{4x-3y+2}{5}\right)^2=8\ \frac{3x+4y-1}{5} \quad \text{.....................(4)}$$



If $\quad 4x-3y+2=0 \quad$ and $\quad 3x+4y-1=0$

be taken as the new axes of x and y respectively, the equation (4) becomes $\quad y^2=8x$.

The vertex of the parabola is the new origin *i.e.* the intersection of $4x-3y+2=0$ and $3x+4y-1=0$, which is the point $(-1/5, 2/5)$.

Thus the given equation represents a parabola, whose axis is $4x-3y+2=0$, whose vertex is the point $(-1/5, 2/5)$ and whose latus-rectum is 8.

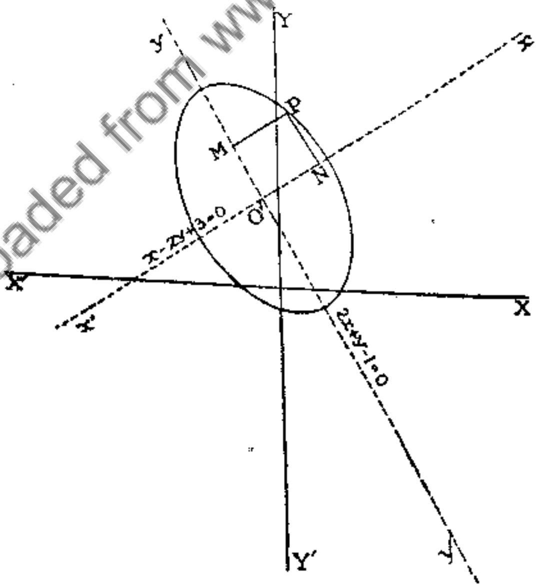
The co-ordinates of focus are given by

$$\frac{x+\frac{1}{5}}{\frac{3}{5}}=\frac{y-\frac{2}{5}}{\frac{4}{5}}=2, \quad i.e. \quad (1, 2).$$

Ex. 4. *Trace the curve*

$$4(x-2y+3)^2+9(2x+y-1)^2=80.$$

We observe that the two straight lines $x-2y+3=0$ and $2x+y-1=0$ are at right angles, so that these be selected for the new axes.



Now $$X = PM = \frac{2x+y-1}{\sqrt{5}}$$

and $$Y = PN = \frac{x-2y+3}{\sqrt{5}}$$

Substituting these in the given equation, we get

$$20Y^2 + 45X^2 = 80$$

or $$\frac{9X^2}{16} + \frac{Y^2}{4} = 1.$$

The centre of the curve is the new origin *i.e.* the intersection of $x-2y+3=0$ and $2x+y-1=0$, which is $(-1/5, 7/5)$.

Thus the given curve is an ellipse, whose centre is $(-1/5, 7/5)$ whose major axis is $x-2y+3=0$ and whose semi-axes are 2 and 4/3 respectively.

Miscellaneous Exercises.

Trace the curve :—

1. $5x^2 - 2xy + 5y^2 - 8x - 8y - 4 = 0.$
2. $2x^2 - 3xy - 2y^2 + 3x - y + 1 = 0.$
3. $x^2 - 4y^2 + x + 4y = 0.$
4. $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0.$
5. $4x^2 + y^2 - 4xy - 10y - 19 = 0.$
6. $x^2 - 4xy - 2y^2 + 10x + 4y = 0.$
7. $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0.$
8. $66x^2 + 24xy + 59y^2 + 312x + 284y + 496 = 0.$
9. $7x^2 - 48xy - 7y^2 + 110x - 20y + 100 = 0.$
10. $x^2 + 4xy + 4y^2 + 7x + 14y + 6 = 0.$
11. $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0.$
12. $5x^2 - 12xy + 10y^2 + 6x - 10y + 6 = 0.$

13. $13x^2+12xy+25y^2+46x+16y+41=0.$ [*Agra 1928*]

14. $20x^2-7xy-6y^2-57x-9y+27=0.$ [*Agra 1929*]

15. $x^2+6xy+9y^2+5x+15y+12=0.$

16. $(x-2y+1)(x+2y-3)=5.$

17. $\dfrac{(3x-4y+12)^2}{4}+\dfrac{(4x+3y-12)^2}{9}=25.$

18. $\sqrt{x}+\sqrt{y}=\sqrt{a}.$

19. $(x+2y-2)^2+4(2x-y+1)^2=45.$

20. Find the eccentricity and the position of the axes of the conic

$$x^2-16xy-11y^2+10x+10y-7=0.$$

21. Shew that the conic

$$30x^2+35y^2=12xy+24x+16y+16$$

has one focus at the origin. Find the equation of the corresponding directrix, the eccentricity and the co-ordinates of the second focus.

22. Prove that the conic

$$9x^2-24xy+41y^2=15x+5y$$

has one extremity of its major axis at the origin, and one extremity of its minor axis on the axis of x. Find the co-ordinates of its centre and foci.

23. Trace

$$(x+2y)^2+12x-6y=0.$$

Find its focus and the equation of the latus-rectum. [*Agra, 1930*]

24. Trace carefully the curve

$$4(2y-x-3)^2-9(2x+y-1)^2=80.$$

Find its eccentricity and the equations to its asymptotes. [*Agra 1932*]

25. Shew that the equation

$$\frac{1}{x+y-a}+\frac{1}{x-y+a}+\frac{1}{y-x+a}=0$$

represents a parabola. Find its focus and directrix.

26. Shew that the curve given by the equation

$$x=at^2+bt+c$$
$$\text{and}\quad y=a't^2+b't+c'$$

is a parabola of latus-rectum $\dfrac{(a'b-ab')^2}{(a^2+a'^2)^{\frac{3}{2}}}$.

CHAPTER X.

SYSTEMS OF CONICS.

10·10. *Number of conditions required to fix a conic.*

The general equation to the conic *viz*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

contains five arbitrary constants*. The conic can not be fixed unless these are definitely evaluated. For this purpose we require five independent relations between them. These can only be furnished by the conditions to be imposed upon the conic. As one condition *generally* leads to one such relation, *five conditions are in general required to fix a conic.*

Illustration 1. *A conic is fixed, if five points through which it passes are known.*

The co-ordinates of every one of these five points when substituted in the general equation to the conic lead to five independent relations between the arbitrary constants there-in, from which these constants can be evaluated.

Illustration 2. *A conic is fixed if four points through which it passes and one straight line which it touches, are known.*

The co-ordinates of the four given points lead to four independent relations between the constants. Condition of tangency of the straight line (Art. 4·32) provides the

*Apparently there are six, but we can divide out by any one of them.

fifth. Between these five all the arbitrary constants can be evaluated.

10·11. In illustration 1 above every one of the five relations between the arbitrary constants will be seen to be linear. They will thus lead to only one set of values for these constants, and therefore only one conic will be found satisfying the given conditions. In illustration 2, however, the first four conditions will give linear relations, but the fifth (condition of tangency) will give a quadratic relation. Evaluation of the arbitrary constants from these five relations will therefore lead to more than one set of values for them, and consequently more than one conic will be found satisfying these conditions.

Thus fixing a conic with the help of a set of specified conditions does not necessarily mean the finding out of one *single* conic satisfying these conditions. It means the determination of *all* such conics, their number depending upon the nature of the relations (between the arbitrary constants) that arise from these conditions.

10·12. The significance of the word *generally* in the first para of Art. 10·10 should be clearly understood. One of the conditions imposed upon the conic may sometimes lead to more than one independent relation between the arbitrary constants. For example, the condition that the centre of a conic is situated at a given point leads to two such relations, obtainable from the two linear equations in x and y giving the co-ordinates of the centre (Art. 4·51). In this case only three more conditions, or even a smaller number (some of these may again lead to more than one relation), are required to fix the conic. Generally speaking, in every case, the number of conditions just enough to give the five independent relations between the arbitrary constants in the general equation to a conic is the number required to fix the conic.

Sometimes the nature of the conic itself assumes some relation or relations between the constants in the general equation to a conic. In such a case, the number of conditions, required to fix it, shall be naturally reduced by the number of these relations. Thus, four conditions will in general be necessary to fix a pair of straight lines ($\Delta=0$), a parabola ($h^2=ab$) or a rectangular hyperbola ($a+b=0$), and only three to fix a circle ($a=b$, $h=0$).

Ex. 1. Find the equation of the conic passing through the points (0, 0), (2, 0), (0, 2), (4, 2) and (2, 4) and determine its nature.

Ex. 2. Determine the equation to the conic passing through the points (0, 0), (10, 0), (5, 3) and symmetrical about the x— axis.

Ex. 3. Find the parabolas passing through the points (0, 2), (0, —2), (4, 0), (—1, 0) and shew that in general two parabolas can be found passing through four given points.

Ex. 4. Find the equation to the conic passing through (0, 5), (5, 0) and symmetrical about both axes.

Ex. 5. Find the equation to a conic passing through the points (0, 0), (2, 0), (5, 2) and having its centre at (3, 4).

10·20. If the number of conditions imposed upon a conic enables us to elliminate only four of the five arbitrary constants in the general equation, the equation to the conic will still contain one of them. For different values of this, we shall obtain different conics. But all of them shall possess certain common properties (as defined by the above conditions), and thus have a sort of family tie between them. They are said to constitute a family or a *system of conics.* In general any equation of the second degree in x and y which contains only one arbitrary constant shall be said to represent a *system of conics,* the arbitrary constant being known as the variable *parameter* of the system.

10·21. Arguing as we did in Art. 3·81, if

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $$S' \equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

be the equations to two conics, then $S+kS'=0$, where k is an arbitrary constant, represents a system of conics passing through the four* common points of $S=0$ and $S'=0$.

The presence of the arbitrary constant k in the equation of the system indicates that conics belonging to the system are capable of satisfying one more condition.

Illustration 1. *One conic of the system can be found passing through any point* (p, q).

Here k is easily seen to be

$$-\frac{ap^2+2hpq+bq^2+2gp+2fq+c}{a'p^2+2h'pq+b'q^2+2g'p+2f'q+c'}.$$

*That $S=0$ and $S'=0$ intersect in four points can be easily seen thus

$$S \equiv ax^2+2x(hy+g)+by^2+2fy+c=0$$
$$S' \equiv a'x^2+2x(h'y+g')+b'y^2+2f'y+c'=0.$$

Elimination of x^2 and x by cross multiplication will lead to a fourth degree equation in y, giving four ordinates for the points of intersection.

Elimination of x^2 in the ordinary way will give a linear equation in x in terms of y. This will lead to one value for x for every one of the four values for y found above, giving the corresponding four abscissae of the points of intersection.

Hence there will be four points of intersection. Of course all of them may be real, or two of them may be real and two imaginary, or all four of them may be imaginary.

Illustration 2. *Three conics of the system can be found which are pairs of straight lines.*

Obviously k will be given by

$$(a+ka')(b+kb')(c+kc') \\ +2(f+kf')(g+kg')(h+kh') \\ -a(f+kf')^2-b(g+kg')^2-c(h+kh')^2=0.$$

Since this is a cubic in k, it will give three values for k, each leading to one pair of straight lines.

The three pairs of straight lines are the pairs passing through the four common points of $S=0$ and $S'=0$ taken two by two.

Ex. 1. How many of the family of conics passing through four points are parabolas ?

Ex. 2. Shew that there is only one rectangular hyperbola passing through the points of intersection of two given conics.

Ex. 3. Shew that all conics through the points of intersection of two rectangular hyperbolas are rectangular hyperbolas.

Ex. 4. Find the equation to the system of conics cutting the axes at the points where $x=1, 3$ and $y=2, 4$.

Ex. 5. If the four points of intersection of the conics $S=0$ and $S'=0$ be concyclic, prove that $\dfrac{a-b}{h}=\dfrac{a'-b'}{h'}$ and the centre of the circle is the point

$$\left(\frac{hg'-h'g}{ah'-a'h}, \frac{hf'-h'f}{ah'-a'h}\right).$$

10·22. What has been said in Art. 10·21 is true whatever be the nature of the conics given by $S=0$ and $S'=0$. Let one of these, say $S'=0$, represent a pair of straight lines α and β, where

$$\alpha \equiv lx+my+n=0$$

$$\beta \equiv l'x+m'y+n'=0$$

Then $S+k\alpha\beta=0$ represents a system of conics passing through the points in which the straight lines $\alpha=0$ and $\beta=0$ intersect the conic $S=0$. Either $\alpha=0$ or $\beta=0$ or both may not meet $S=0$ in real points. $S+k\alpha\beta=0$ can still be regarded as a conic passing through the corresponding imaginary points of intersection.

$\alpha=0$ and $\beta=0$ can intersect $S=0$* in real points in six different ways as shown in the following figures :—

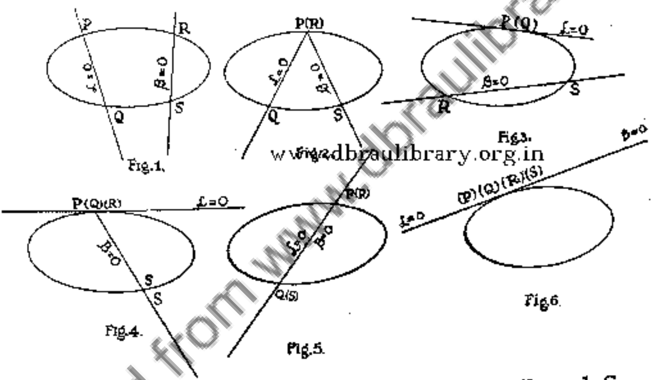


Fig.1.

Fig.2.

Fig.3.

Fig.4.

Fig.5.

Fig.6.

In *Fig. 1,* $S+k\alpha\beta=0$ cuts $S=0$ in P, Q, R, and S.

In *Fig. 2,* P and R coincide *i.e.,* $\alpha=0$ and $\beta=0$ meet on the conic itself. $S+k\alpha\beta=0$ touches $S=0$ at P (R).

In *Fig. 3,* P and Q coincide *i.e.*; $\alpha=0$ touches $S=0$. $S+k\alpha\beta=0$ also touches $S=0$ at $P(Q)$ and $\alpha=0$ is the common tangent.

In *Fig. 4,* P, Q and R coincide *i.e.,* $\alpha=0$ touches $S=0$ at a point where $\beta=0$ meets it. $S+k\alpha\beta=0$ and $S=0$ have not only the common tangent $\alpha=0$, but also a common chord $\beta=0$.

*The conic $S=0$ in all the following figures has been taken to be the ellipse. This is immaterial. It could be any other.

In *Fig.* 5, P coincides with R, and Q with S *i.e.* the straight lines $\alpha=0$ and $\beta=0$ coincide. $S+k\alpha\beta=0$ becomes $S+k\alpha^2=0$ and touches $S=0$, at the points where $\alpha=0$ cuts it. Conics touching in such a manner are said to have a double contact.

In *Fig.* 6, P, Q, R and S, all coincide. $\alpha=0$ and $\beta=0$ coincide and touch $S=0$. $S+k\alpha\beta=0$ becomes $S+kT^2=0$ where $T=0$ is the equation to the tangent to $S=0$ and has a four point contact with $S=0$.

10·23. If $S=0$ represents a conic and $U=0$ a straight line, the equation $S+kU=0$ represents a conic passing through the two points in which $S=0$ is intersected by $U=0$. From Art. 7·47, the two conics are seen to have their asymptotes (real or imaginary) parallel, and hence their axes, in the same directions. When $U=0$ is a tangent to $S=0$, it is only a special case of the above, the conic $S+kU=0$ in that case touches the conic $S=0$ where $U=0$ touches it.

Also from Art. 7·48, it is clear that $S+k=0$ represents a conic having the same asymptotes as the conic $S=0$.

10·24. The case of *Fig.* 5, furnishes a very handy method for finding the equation to the pair of tangents to any conic from a given point.

Let the given conic be $S=0$, the given point (x_1, y_1), and let $U=0$ be the chord of contact of tangents drawn from the point (x_1, y_1). Also let S_1 and U_1 denote the transformed S and U when x and y therein are replaced by x_1 and y_1.

Then the required pair of tangents is a conic

(1) having double contact with the conic $S=0$ at the points where the latter is cut by $U=0$,

and (2) passing through the point (x_1, y_1).

From (1) its equation is $S+kU^2=0$.

From (2) $\quad k=-\dfrac{S_1}{U_1^2}=-\dfrac{1}{S_1} \quad [S_1=U_1]$.

Hence the required equation is $SS_1=U^2$. (See Art. 4·41)

10·25. If alongside with $S'=0$ (Art. 10·22), $S=0$ also represents a pair of straight lines, say

$$\gamma \equiv px+qy+r=0$$

$$\delta \equiv p'x+q'y+r'=0.$$

$\gamma\delta+k\alpha\beta=0$ is the equation to the system of conics passing through the four points in which $\gamma=0$ and $\delta=0$ intersect $\alpha=0$ and $\beta=0$ respectively.

It should be carefully noted that the conic $\gamma\delta+k\alpha\beta=0$ does not pass through the point of intersection of $\gamma=0$ and $\delta=0$ nor through that of $\alpha=0$ and $\beta=0$. It passes through the four points in which $\gamma=0$ and $\delta=0$ respectively intersect $\alpha=0$ and $\beta=0$ *i.e.* it circumscribes the quadrilateral having $\alpha=0$, $\beta=0$ and $\gamma=0$, $\delta=0$ as pairs of opposite sides.

Ex. 1. Write down the general equation of the conic passing through the points $(0, -2)$, $(2, 0)$, $(-1, 0)$ and $(3, 4)$.

From it determine the parabolas through these points and also the conic passing the fifth point $(1, 1)$.

Ex. 2. Find the equation of the conic through the points $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(0, -2)$, $(2, 0)$, $(2, 1)$, and trace it.

10·30. *To find the equation to a conic touching two given straight lines.*

The tangency of two straight lines imposes two conditions upon the arbitrary constants in the general equation to a conic. Hence the required equation will still contain three arbitrary constants.

Let the two given tangents be

$$y=m_1x+c_1$$

$$\text{and} \quad y=m_2x+c_2,$$

and let the chord of contact be $\frac{x}{a}+\frac{y}{b}=1$, where a and b are arbitrarily chosen.

Then the required equation is (Art. 10·22 Fig. 5.)

$$(y-m_1x-c_1)(y-m_2x-c_2)+k\left\{\frac{x}{a}+\frac{y}{b}-1\right\}^2=0. \quad ...(1)$$

k being the third arbitrary constant besides a and b. The equation to the conic referred to the tangents as the axes of co-ordinates is evidently

$$xy+k\left\{\frac{x}{a}+\frac{y}{b}-1\right\}^2=0 \quad ....(2)$$

Ex. CA and CB are the transverse and conjugate semi-diameters of a hyperbola, and a parabola is drawn to touch these axes at A and B. Prove that one of the asymptotes of the hyperbola is parallel to the axis of the parabola, and the other asymptote is parallel to chords bisected by the first asymptote.

10·31. *To find the equation to a parabola touching the two axes of co-ordinates.*

Equation (2) above will give a parabola if

$$\left\{1+\frac{2k}{ab}\right\}^2=\frac{4k^2}{a^2b^2} \qquad (H^2=AB).$$

$$or \quad k=-\frac{ab}{4}.$$

With this value for k, the equation becomes

$$\frac{x}{a}+\frac{y}{b}-1= \pm 2\sqrt{\frac{xy}{ab}}$$

or $$\left\{\sqrt{\frac{x}{a}} \pm \sqrt{\frac{y}{b}}\right\}^2=1$$

or $$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,$$

where the radicals can have either sign prefixed to them.

Ex. 1. Find the condition that the straight line $\frac{x}{\alpha}+\frac{y}{\beta}=1$ should touch the parabola which itself touches the axes at the points $(a, 0)$ and $(0, b)$.

10·32. *To find the equation to a system of conics touching four given straight lines.*

Let two of these straight lines be the axes of co-ordinates, and the other two be given by $y=m_1x+c_1$ and $y=m_2x+c_2$.

Then if $\frac{x}{a}+\frac{y}{b}=1$ be the equation to the chord of contact of the axes, the equation to the conic touching these is

$$xy+k\left\{\frac{x}{a}+\frac{y}{b}-1\right\}^2=0 \quad \text{...(1)} \quad \text{Art. 10·31.}$$

The conditions that $y=m_1x+c_1$ and $y=m_2x+c_2$ are tangents to this conic can be easily seen to be (Art. 4·32).

$$c_1^2(4k+ab)+4kc_1(am_2-b)-4km_1ab=0 \quad \text{.........(2)}$$

$$c_2^2(4k+ab)+4kc_2(am_2-b)-4km_2ab=0 \quad \text{.........(3)}$$

Eliminating a and b between (1), (2) and (3), we get the equation to the required conic.

Miscellaneous Exercises.

1. Find the equation to the system of conics touching the axes at $x=3$ and $y=5$. Shew that of these, two are parabolas.

2. Find the equation to a conic through the points where the axes meet the conic $3x^2+4xy+y^2-y+2=0$ and through the point $(-1, 1)$.

3. Find the locus of the centre of the conics passing through the intersections of the two conics $S=0$ and $S'=0$.

4. A conic has double contact with the parabola $y^2=4ax$. If the chord of contact passes through the vertex and the conic passes through the focus, prove that the locus of the centre of the conic is the parabola $y^2=a(2x-a)$.

5. Shew that the equation of the circle having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum is $x^2+y^2-2ae^3x=a^2(1-e^2-e^4)$, where e is the eccentricity of the ellipse.

6. Find the locus of the centres of conics touching two given straight lines at two given points.

7. TP and TQ are tangents from $T\ (\alpha, \beta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the equation to the parabola having double contact with the ellipse at P and Q is $(x\beta-y\alpha)^2+2b^2\alpha x+2a^2\beta y=b^2\alpha^2+a^2\beta^2+a^2b^2$.

8. A series of rectangular hyperbolas are drawn each passing through a given point and touching a given line at a given point. Prove that their centres lie on a circle.

9. A hyperbola touches the axis of y at the origin, and the line $y=7x-5$ at the point $(1, 2)$. One of its asymptotes is parallel to the axis of x. Find the equation of the curve.

10. A system of conics have the same focus and directrix. Find the locus of points the tangents at which are parallel to a given line.

11. Prove that the general equation to the ellipse, having double contact with the circle $x^2+y^2=a^2$ and touching the axis of x at the origin is

$$c^2x^2+(a^2+c^2)\,y^2=2a^2cy.$$

12. Find the equation of the parabola which touches the conic $x^2+xy+y^2-2x-2y+1=0$ at the points where it is cut by the line $x+y+1=0$.

Determine the equation of the axis and the co-ordinates of the focus of this parabola. [*Lucknow 1935*]

13. Find the conditions that the chords of intersection of the conics

$$ax^2+2hxy+by^2=1$$

$$\text{and}\quad a'x^2+2h'xy+b'y^2=1$$

may be at right angles.

14. If the equations $L_1=0$, $L_2=0$ and $L_3=0$ represent straight lines, and $S_1=0$, $S_2=0$ and $S_3=0$ represent conics, interpret the following equations

(*i*) $aL_1L_2+bL_2L_3+cL_3L_1=0$,

(*ii*) $aS_1+bS_2+cS_3=0$,

where a, b, c are constants. [*Lucknow 1937*]

15. Find the conic passing through the points of intersection of $y^2+yx-x^2=1$ and $x^2-xy+5y^2+2x-8=0$ and the origin.

10·40. *Confocal Conics* (*Central*).

Conics having the same foci are said to be *confocal*.

All such central conics have obviously the same centre and their axes lie along same lines. Hence the distance between the centre and foci is the same for all of them. Thus if $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ is a given central conic and $\frac{x^2}{\alpha'^2}+\frac{y^2}{\beta'^2}=1$ a conic confocal with it, then

$$\alpha e=\alpha' e'$$

$$\text{or} \quad \alpha^2-\beta^2=\alpha'^2-\beta'^2 \qquad \ldots\ldots\ldots\ldots(1)$$

If in (1) α'^2 be put as $\alpha^2+\lambda$, where λ is an arbitrary constant, β'^2 can be put as $\beta^2+\lambda$, and the general equation to a conic confocal with the given conic, sometimes called the fundamental conic, is seen to be

$$\frac{x^2}{\alpha^2+\lambda}+\frac{y^2}{\beta^2+\lambda}=1. \qquad \ldots\ldots\ldots\ldots(2)$$

As equation (2) contains only one arbitrary constant λ, the confocals constitute a *System of Conics* (Art. 10·20) with λ as the variable *parameter*.

Ex. 1. Prove that the conics $x^2-y^2-4x+2y+2=0$ and $x^2+3y^2-4x-6y+4=0$ are confocal.

Ex. 2. Find the conics confocal with $x^2+2y^2=2$, which pass through the point (1, 1).

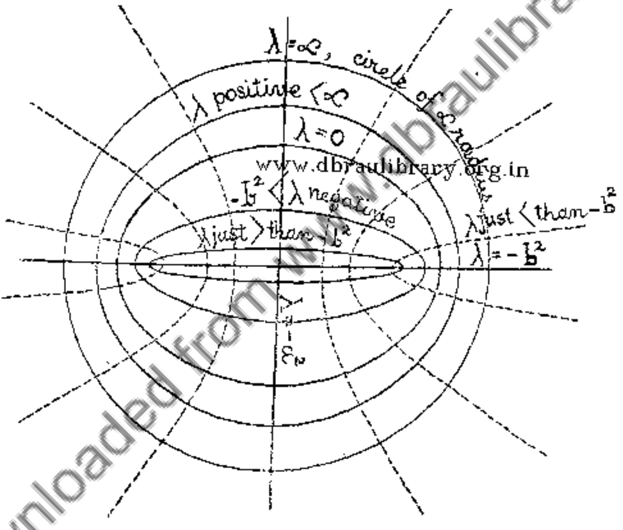
10·41. *To trace the Confocals given by*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Assume $a>b$.

(1) For all positive values of λ and for such negative values as are numerically less than b^2, the equation represents an ellipse.

For very large positive values of λ, the semi-axes of the ellipse are very large and it is almost a circle. For $\lambda = \infty$, it is a circle of infinite radius. As λ travels from ∞ to $-b^2$, the ellipse gradually flattens, till for λ very nearly equal to $-b^2$, the minor axis is very small, the major axis is very nearly $\sqrt{a^2-b^2}$, and the conic approximates to the $x-$axis lying between the foci.



(2) For $\lambda = -b^2$, the conic degenerates into the $x-$axis.

(3) For negative values of λ lying between $-b^2$ and $-a^2$, the conic is a hyperbola gradually opening out as λ travels from $-b^3$ to $-a^2$. For λ very nearly equal to $-b^2$, the conjugate axis is very small and the two branches of the curve

approximate to portions of the x-axis lying beyond the foci on the two sides of the centre. For λ very nearly equal to $-a^2$, the transverse axis is very small, and both the branches of the curve approximate to the y-axis.

(4) For $\lambda = -a^2$, the conic degenerates into the y-axis.

(5) For negative values of λ numerically less than a^2, the equation represents an imaginary ellipse *i.e.*, ceases to represent any real curve.

10·42. *To shew that through any point in the plane of a given central conic, there pass two conics confocal with it, one an ellipse and the other a hyperbola.*

Let the given conic be the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Any conic confocal with the given conic is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

If this is to pass through a point (x_1, y_1),

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1.$$

$$or \quad (a^2+\lambda)(b^2+\lambda)-x_1^2(b^2+\lambda)-y_1^2(a^2+\lambda)=0.$$

The above equation being a quadratic in λ gives two values of λ. So there are two conics of the system passing through the point (x_1, y_1).

Also the left hand expression in the above equation is positive, negative or positive according as λ is ∞, $-b^2$ or $-a^2$. Hence one root of λ lies between ∞ and $-b^2$ and the other between $-b^2$ and $-a^2$. For the former $a^2+\lambda$ and $b^2+\lambda$ have both the same sign and for the latter opposite signs. Thus one of the two confocals through (x_1, y_1) is an ellipse, and the other a hyperbola.

If we had taken the given conic as the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ instead of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the proposition could be argued on similar lines.

Cor. *If the point (x_1, y_1) be situated on the fundamental conic itself, only one more confocal can pass through it, which is a hyperbola if the fundamental conic is an ellipse and vice-versa.*

Ex. Shew that the equation to the hyperbola confocal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point 'α' on it, is $\dfrac{x^2}{\cos^2\alpha}-\dfrac{y^2}{\sin^2\alpha}=a^2-b^2$

10·43. *To shew that a given straight line will always be touched by one and only one conic of a central confocal system.*

Let the system of confocals be given by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and let the given straight line be

$$lx+my=n \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If (2) is a tangent to (1)

$$(a^2+\lambda)\, l^2+(b^2+\lambda)\, m^2=n^2 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) being linear in λ gives only one value of λ. Hence there shall always be one and only one confocal of the system touching the given line.

10·44. *To shew that two conics of a central confocal system cut at right angles at every point of their intersection.*

Let

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \text{and} \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

be two such conics and (x_1, y_1) a common point.

Then $$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1 \quad \ldots\ldots\ldots\ldots(1)$$

and $$\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}=1 \quad \ldots\ldots\ldots\ldots(2)$$

Subtracting (2) from (1)

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \quad \ldots\ldots(3)$$

Also tangents to the two conics at (x_1, y_1) are

$$\frac{xx_1}{a^2+\lambda_1}+\frac{yy_1}{b^2+\lambda_1}=1 \quad \ldots\ldots\ldots\ldots(4)$$

and $$\frac{xx_1}{a^2+\lambda_2}+\frac{yy_1}{b^2+\lambda_2}=1 \quad \ldots\ldots\ldots\ldots(5)$$

The condition of their perpendicularity is

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \quad \ldots\ldots(6)$$

Same as (3) which we know to be true. Hence the proposition.

10·45. *To express the Cartesian Co-ordinates of any point P in the plane of the fundamental conic*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in terms of the parameters λ_1 and λ_2 of its two confocals passing through that point.

Let the point P be (x_1, y_1).

Then $$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1$$

and $$\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}=1$$

whence

$$\frac{x_1^2}{\dfrac{1}{b^2+\lambda_2}-\dfrac{1}{b^2+\lambda_1}}=\frac{y_1^2}{\dfrac{1}{a^2+\lambda_1}-\dfrac{1}{a^2+\lambda_2}}$$

$$=\frac{1}{\dfrac{1}{(a^2+\lambda_1)(b^2+\lambda_2)}-\dfrac{1}{(b^2+\lambda_1)(a^2+\lambda_2)}}$$

On simplification

$$x_1^2=\frac{(a^2+\lambda_1)(a^2+\lambda_2)}{a^2-b^2}, \quad y_1^2=\frac{(b^2+\lambda_1)(b^2+\lambda_2)}{b^2-a^2}.$$

The position of the point P is thus definitely fixed with the help of the parameters λ_1 and λ_2. These are called the *Elliptic Co-ordinates* of the point.

10·50. *Confocal Parabolas.*

Parabolas having the same focus and axis are said to be *Confocal.*

The equation to any parabola referred to its focus as the origin and its axis as the axis of x is $y^2=4\lambda(x+\lambda)$, where 4λ is its latus-rectum.

For different values of λ the above equation represents a system of *confocal Parabolas.*

Proceeding as in the case of the central conics, the student can easily verify the following properties for this system :

(1) Through any point in the plane of a given parabola there pass two parabolas confocal with it, with their concavities in opposite directions.

(2) A given straight line will be touched by one and only one confocal of the system.

(3) Confocal parabolas will cut at right angles at every point of their intersection.

(4) The Cartesian Co-ordinates x_1 and y_1 of any point P in the plane of the fundamental parabola $y^2=4a(x+a)$, when expressed in terms of the parameters λ_1 and λ_2 of the confocal parabolas passing through it are

$$x_1= -(\lambda_1+\lambda_2) \quad \text{and} \quad y_1=\sqrt{-4\lambda_1\lambda_2}.$$

λ_1, λ_2 are called the *parabolic co-ordinates* of the point.

Ex. 1. The locus of the pole of the straight line $lx+my=1$ with respect to the confocal parabolas $y^2=4\lambda(x+\lambda)$ is the straight line $mx-ly+m/l=0$.

Ex. 2. Shew that the locus of the intersection of two perpendicular tangents, one to each of two confocal parabolas, is a straight line perpendicular to the axis.

Miscellaneous Exercises.

1. Shew that the locus of the intersection of the tangents to a variable ellipse of a confocal family at points having given eccentric angles is a hyperbola.

2. Two mutually perpendicular straight lines are so related with respect to a conic that each passes through the pole of the other. Shew that they are similarly related with respect to any confocal conic.

3. Shew that the general equation of conics whose foci are the given points (a, b) and (a', b') is

$$\{(x-a)(b-b')-(y-b)(a-a')\}^2 + 2\lambda\{(x-a)(x-a')+(y-b)(y-b')\}-\lambda^2=0.$$

4. Shew that the locus of the points of contact of tangents drawn in a fixed direction to a system of confocal conics is a rectangular hyperbola.

5. Shew that, if tangents are drawn to a system of confocal conics from a fixed point on the major axis, the locus of the points of contact is a circle whose centre is on the major axis.

6. The locus of points such that two tangents drawn from them, one to each of two confocals, are at right angles, is a circle concentric with the two confocals.

7. Shew that the parameter λ, of the confocal conic which passes through a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is equal to CD^2, where CD is the semi-diameter of the ellipse conjugate to CP.

8. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from any point on a confocal hyperbola. If 2α is the angle between them, shew that $\sin\alpha$ varies inversely as CD, the semi-diameter conjugate to CP of the confocal ellipse through P.

9. The two conics $a_1x^2+2h_1xy+b_1y^2=1$ and $a_2x^2+2h_2xy+b_2y^2=1$ can be placed so as to be confocal, if

$$\frac{(a_1-b_1)^2-4h_1^2}{(a_1b_1-h_1^2)^2}=\frac{(a_2-b_2)^2-4h_2^2}{(a_2b_2-h_2^2)^2}$$

10. If 2α is the angle between the tangents from a point P to the parabola $y^2=4a(x+a)$, and b and $-c$ are the parameters of the two confocals through P, shew that $\tan^2\alpha=\dfrac{c+a}{b-a}$.

11. If through any point on the Director Circle of a given ellipse, two confocals are drawn whose parameters are λ_1 and λ_2, prove that the sum of the parameters is constant.

12. Two tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, at points whose eccentric angles are θ and ϕ, intersect in a point on a confocal ellipse whose parameter is λ. If the tangents are at right angles, prove that

$$\cos(\theta-\phi)=\frac{a^2b^2-\lambda^2}{a^2b^2+\lambda^2}.$$

13. If the two confocal ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and $$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

be cut by the straight line

$$x\cos\theta+y\sin\theta=p,$$

and if T and T' be the poles of this line with respect to the two ellipses, prove that $TT'=\lambda/p$.

14. If λ, μ are the values of the parameters for the parabolas of the confocal system $y^2=4\lambda(x+\lambda)$ which pass through P, prove that the angle ϕ between the tangents from P to the parabola $y^2=4k(x+k)$ is given by $tan^2 \phi = -(\lambda-k)(\mu-k)$.

15. If ϕ is the angle between the tangents, to the conic $\lambda=k$ of the confocal system $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, from the point whose elliptic co-ordinates referred to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are λ, μ, shew that

$$tan^2 \phi = \frac{-4(\lambda-k)(\mu-k)}{(\lambda+\mu-2k)^2}.$$

16. From a fixed point P, pairs of tangents PQ and PQ' are drawn to each of a system of confocal conics. Prove that the circles PQQ' form a coaxal system.

ANSWERS.

Art. 1·3 (page 6)

1. $(3, 0)$, $(\sqrt{61}, \pi + \tan^{-1} 6/5)$, $(5, \pi)$, $(4, 3\pi/2)$ and $(\sqrt{5}, \pi - \tan^{-1} 1/2)$.

2. $(4, 4\sqrt{3})$; $5\sqrt{3}/2, 5/2$; $(-9/\sqrt{2}, 9/\sqrt{2})$; $1, -\sqrt{3}$, $(-1/\sqrt{2}, -1/\sqrt{2})$ and $(-2\sqrt{3}, 2)$.

3. (i) $r(\sin\theta - m\cos\theta) = c$.
   (ii) $r = a$.
   (iii) $r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$.
   (iv) $r(1 - \cos 2\theta) = 8a\cos\theta$.
   (v) $r\cos^3\theta = a$.
   (vi) $r^2(1 + \sec 2\theta) + 2a^2 = 0$.
   (vii) $r\cos\theta = a(1 - \cos 2\theta)$.
   (viii) $r^3\sin^3\theta = (r\cos\theta - a)^2 (r\cos\theta - b)$.

4. (i) $x\cos\lambda + y\sin\lambda = p$.
   (ii) $x^2 + y^2 = 2ax$.
   (iii) $(x^2 + y^2)^3 = a^2x^2y^2$.
   (iv) $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$.
   (v) $x^2 + y^2 = a(x + y)$.
   (vi) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
   (vii) $x = a$.
   (viii) $(x^2 + y^2 - 2a^2) = x^2(x^2 + y^2)$.

Art. 2·12 (page 11)

. (i) $r\cos\theta = a$.
   (ii) $r\sin\theta = a$.

. $p = r\cos(\theta + \alpha)$.

Art. 2·13 (page 12)

1. (i) $\dfrac{a}{r}=\sin\theta-\cos\theta/3\sqrt{3}.$

(ii) $\dfrac{\sin 60}{r}+\dfrac{\sin(45-\theta)}{7}+\dfrac{\cos(15-\theta)}{3}=0.$

(iii) $\dfrac{\sin 30}{r}+\dfrac{\cos(30+\theta)}{a}+\dfrac{\sin(30-\theta)}{2a}=0.$

2. $\left\{\dfrac{2r_1r_2\cos\dfrac{\theta_1-\theta_2}{2}}{r_1+r_2},\ \dfrac{\theta_1+\theta_2}{2}\right\}.$

Art. 2·4 (pages 13—15)

3. $y=x+1$; $x+2y=8$; $3x+y=9$.

4. $77x+99y=9$; $16x+28y+1=0$; $x-y=1$.

5. $x=3$; $x(\sqrt{13}-1)+y(5+\sqrt{13})=6\sqrt{13}$;
$y(5+\sqrt{13})=x(1+\sqrt{13})$.

Art. 2·7 (page 17)

1. (i) $(x+y)(x+2y)(x+3y)=0.$

(ii) $(x^2-xy+y^2)(x^2-4xy+y^2)=0.$

(iii) $(x-y)(1-xy)=0.$

2. $x^2+8xy-11y^2-44x+94y=191.$

Art. 2·84 (page 19)

1. $2x+3y+4=0$ and $3x+4y+5=0$.

(i) $6x^2+17xy+12y^2=0.$

(ii) $12x^2-17xy+6y^2=0.$

Art. 2·85 (page 20)

1. $38/17\sqrt{5}$.　　2. $(4/5,\ 0)$.

Art. 2·87 (page 22)

1. $k=3$ and $p=-40$.

2. (i) $4x^2+17xy+15y^2-42x-77y+98=0.$

(ii) $15x^2-17xy+4y^2+4x+y-3=0.$

3. $6x^2+xy-y^2+53x+9y+112=0.$

Art. 2·9 (page 24)

3. $c^2=a^2m^2+b^2$.

Examples (pages 24—26)

1. $x-y=1$ 4. $5/2$.

Art. 3·1 (page 28)

1. (i) $r^2-2\sqrt{2}ar\cos\theta+a^2=0$, a being a variable.

(ii) $r^2-2rR\cos\theta+r^2=a^2$, a being the fixed distance of the given line.

2. $r^2-2ar\operatorname{cosec}\alpha\cos(\theta-\alpha)+a^2\cot^2\alpha=0$. $r=2a\sin\theta$.

4. $r^2-r(A\cos\theta+B\sin\theta)+\frac{1}{4}(A^2+B^2)=a^2$.

Art. 3·62 (page 33)

1. $x^2+y^2+gx+fy+c/2=0$.

Art. 3·74 (page 35)

1. (i) $10x=7y$. (ii) $31x-35y+25=0$.

2. $ax=by$; $2ab/\sqrt{a^2+b^2}$.

Art. 3·75 (page 36)

1. (i) $(-16/5, 17/15)$. (ii) $(31/7, -10/7)$.

2. $(33/4, 13)$; $\sqrt{2641/4}$.

Art. 3·81 (page 37)

1. $(x-1)^2+(y-1)^2-27+\lambda(x^2+y^2-25)=0$.

2. $x^2+y^2+102x+48y-23=0$.

3. $x^2+y^2+2\dfrac{g'c-gc'}{c-c'}x+2\dfrac{f'c-fc'}{c-c'}y=0$.

4. $3x^2+3y^2+8x-4y=0$.

5. $x^2+y^2+2gx+2fy+c+\lambda(x-y)=0$.

6. $x^2+y^2-16+\lambda(3x-5y-7)=0$.

Art. 3·83 (page 39)

1. $4(x^2+y^2\pm 5x)+29=0.$

Art. 3·91 (pages 40—41)

2. (i) $(-1, 1)$ and $(1/5, 8/5)$.
   (ii) $(1, 0)$ and $(-1, 0)$.

Art. 3·92 (page 42)

3. (i) Imaginary ; $(2, 3)$ and $(4, 9)$.
   (ii) Circles touching ; $(3, 4)$.
   (iii) $(-9, 0)$ and $(-9, 6)$; Imaginary.

Art. 3·93 (page 42)

1. (i) $(x^2+y^2+a^2)(1+\lambda)+2ax(1-\lambda)=0.$
   (ii) $\{(x-2)^2+(y-3)^2\}+\lambda\{(x-4)^2+(y-9)^2\}=0.$
   (iii) $\{(x+1)^2+(y-2)^2\}+\lambda\{(x-3)^2+(y+5)^2\}=0.$

Art. 3·94 (page 43)

1. $x^2+y^2+2\lambda x+2(\lambda+1)y+3(2\lambda-1)=0,$
2. $x^2+y^2-4x-2y+2=0.$

Examples (pages 45—46)

2. $7x^2+7y^2-41x-17y+64=0,$
   $3x^2+3y^2-20x-8y+37=0,$
   $2x^2+2y^2-19x-7y+47=0.$

Miscellaneous Exercises (pages 46—49)

1. $x^2+y^2-8x-10y+16=0.$
   $x^2+y^2+72x-250y+1296=0.$
2. $x^2+y^2=(5\pm 2\sqrt{2}).$
3. $2x-2y=3.$
4. $5x-12y+10=0,\ 5x-12y+75=0,\ x^2+y^2-6y+5=0.$
6. $(0, -8)$; $\sqrt{221}.$

7. $x^2+y^2-6x+6y+9=0.$

8. A circle passing through the point 0.

9. A circle passing through the point 0.

12. $9(x^2+y^2)=(x^2-y^2+2x+y)^2.$

14. $x-1=0$; $y-2=0$; $4x-3y=10$; $3x+4y=5.$

18. $x^2+y^2-4x-2y+3=0.$

22. $\sqrt{4c^2-2(a-b)^2}.$

24. $(x\pm3)^2+(y\pm4)^2=49.$

25. $x^2+y^2-(b-6)x-4y=0.$

Art. 4·25 (page 61)

1. $\theta=tan^{-1}\ 1/2.$ 2. $\theta=tan^{-1}\ 3/2.$

3. $\theta=tan^{-1}\ 4/3.$

Art. 4·26 (page 64)

1. (i) Hyperbola.
   (ii) Pair of imaginary straight lines.
   (iii) Circle.
   (iv) Pair of straight lines.
   (v) Parabola.
   (vi) Rectangular hyperbola.
   (vii) Pair of coincident straight lines.
   (viii) Ellipse.

Art. 4·30 (page 65)

2. 169/121.

Art. 4·32 (page 67)

1. $m=11,\ 81/19.$
   $y=11x-9,\quad 19y=81x+85.$

2. $y+x=y\pm4\sqrt{2}.$

4. When $f^2=bc$,
one root of m is ∞ *i.e.* y axis is a tangent.

Art. 4·41 (page 69)

1. $121x^2-156xy+24y^2+70x+60y-95=0$;
$Tan^{-1}(\cdot 78)$.

Art. 4·43 (page 71)

1. $3(x^2+y^2)+2(x+y-1)=0$.

Art. 4·50 (page 72)

2. $Tan^{-1}\frac{1}{2}$, $2\sqrt{5}$.

Art. 4·51 (page 73)

1. (*i*) $(2, 2)$; (*ii*) $(1, 1)$, (*iii*) $(\frac{1}{2}, 1)$.

Art. 4·52 (page 74)

1. $3x+3y+2=0$.

2. $3x-5y+7=0$.

Art. 4·61 (page 76)

1. (*i*) $(1, 1)$; (*ii*) $(1, \frac{1}{2})$.

Miscellaneous Exercises (pages 76—78)

3. $(a, 0)$.

5. $\alpha^2(ax+hy+g)^2+\beta^2(hx+by+f)^2=(gx+fy+c)^2$.

Art. 5·41 (page 83)

1. $(4a/m^2, -4a/m)$.

7. $27ay^2=(2x-a)(x-5a)^2$.

Miscellaneous Exercises (pages 84—88)

1. $y^2+20x-6y-71=0$; $x=4$.

2. $(x+2)^2=y-3$.

12. $y^2=2ax+k^2x^2$.

14. $\{(h-6a),\ -k/2\}$.

24. $(x+a)^2=y^2+4a^2$.

Examples (pages 96—99)

1. $(x-5)^2+4(y-7)^2=100$; $(5+5\sqrt{3},\ 7)$, $(5-5\sqrt{3},\ 7)$.

2. $x^2/a^2+y^2/b^2=\sec^2\alpha$.

5. $k^2y^2-2xy=b^2k^2$, where k is the constant.

11. $c^2(x^2/a^2+y^2/b^2)=a^2b^2(x^2/a^4+y^2/b^4)\times(x^2/a^2+y^2/b^2-1)$.

13. $2ab\sqrt{\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}-1\right)\left(\dfrac{h^2}{a^4}+\dfrac{k^2}{b^4}\right)}\Bigg/\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}$.

15. $\alpha^2x^2/a^2+\beta^2y^2/b^2=1$.

19. $(a/2,\ 0)$.

Art. 6·62 (page 103)

2. $a^2ly-b^2mx=0$.

4. $x^2/a^2+y^2/b^2=1/2$.

Miscellaneous Exercises (pages 104—107)

2. $ax\pm by=\pm\sqrt{a^4+a^2b^2+b^4}$.

3. $a^2=2b^2$.

7. $a^2x^2+b^2y^2=a^4+b^4$.

10. $(x^2/a^2+y^2/b^2)^2=x^2+y^2/a^2+b^2$.

11. $(x^2/a^2+y^2/b^2)^2(1/a^2+1/b^2)=x^2/a^4+y^2/b^4$.

15. $(x^2/a^2+y^2/b^2+2)^2=9\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$.

16. $x/a-4y/b+2=0$.

23. $x^2/a^2+y^2/b^2-1-\lambda\{(a^2-b^2)xy+b^2kx-a^2hy\}=0$,
where $\lambda=\pm2/ab(a^2-b^2)$.

Examples (pages 112—115)

1. $7x^2+12xy-2y^2-2x+14y-22=0.$
2. $[(x'^2+y'^2-a^2)/(x'^2-a^2)]^{\frac{1}{2}}.$
3. $(x-3)^2/16-(y+2)^2/4=1.$
4. $(x-4)^2-(y-5)^2=16$; $x=4\pm2\sqrt{2}$; $\{4(1\pm\sqrt{2}), 5\}$.
9. $p^2=m^2(a^2m^2-b^2)$ and $c=p/m.$

16. $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)(a^2+b^2)^2=(a^2-b^2)(x^2+y^2)\left(\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)^2.$

17. $x^2/a^4+y^2/b^4=\dfrac{1}{a^2}-\dfrac{1}{b^2}.$

19. $(l^2+m^2)/n^2=\dfrac{1}{a^2}-\dfrac{1}{b^2}$, $lx+my+n=0$ being the chord.

Exercises (pages 121—122)

1. (*i*) $y-3=0$; $x+4=0.$
   (*ii*) $x-y=0$; $5x+3y-4=0.$
   (*iii*) $(x-y-1)=0$; $2x+3y-2=0.$
2. $9y^2-72y-16x^2=0.$
3. $2x^2-xy-6y^2-2x+25y-46=0.$

Exercises (pages 127—128)

1. $5/\sqrt{3}.$
2. $3x^2-5xy-2y^2+5x+11y-16=0.$

Miscellaneous Exercises (pages 130—133)

2. $x\pm y=\sqrt{(a^2-b^2)}.$
14. $2\sqrt{2}l$; $\sqrt{2}e$, where l and e are the semi-latus rectum and eccentricity of the given conic.

Art. 8·12 (page 136)

2. $l=er\cos(\theta-\alpha).$
3. $125/24.$

Art. 8·31 (pages 139—140)

2. $\tan^{-1}(1-e\cos\alpha/e\sin\alpha)$.

4. $\theta=\dfrac{\alpha+\beta}{2}$.

Art. 8·5 (page 142)

2. $l/r=\cos\alpha-e\cos\theta$.

Art. 9·20 (pages 146—147)

1. $(-2/7,\ -3/7)$

2. $7x^2-9xy+4y^2+6=0$.

3. 3.

Art. 9·21 (page 148)

1. (i) $\tan^{-1}(-1/3)$, $\tan^{-1}3$.
   (ii) $\tan^{-1}(\frac{1}{2})$, $\tan^{-1}(-2)$.

2. (i) $x-y+3=0$; $x+y-1=0$.
   (ii) $3x-2y+1=0$; $2x+3y-2=0$.

Art. 9·22 (page 149)

1. $8, 8/\sqrt{3}$.

Art. 9·24 (page 152)

2. (i) $(1, 1)$, $(-5/3, -1/3)$.
   (ii) $(-1, 3)$, $(3, 1)$.
   (iii) $(-4, -4)$, $(-1, -1)$.

Art. 9·31 (page 154)

1. (i) $x-y+3=0$, $(-3/2, 3/2)$, $\sqrt{2}$.
   (ii) $2x+y+3=0$, $(-2, 1)$, $\sqrt{5}$.

2. $(3/5, 4/5)$.

3 $36x-36y+77=0$, $(-23/72, -31/72)$.

Miscellaneous Exercises (pages 159—161)

1. Ellipse ; centre (1, 1), Semi-axes $\sqrt{3}$ and $\sqrt{2}$, Major axis inclined at 45° to the axis of x.
2. Two straight lines ; $2x+y+1=0$ and $x-2y+1=0$.
3. Hyperbola ; centre $(-\frac{1}{2}, \frac{1}{2})$, Semi-axes $\sqrt{3}/2$ and $\sqrt{3}/4$, asymptotes $2x+4y-1=0$ and $2x-4y+3=0$.
4. Ellipse ; centre $(-1, 1)$, Semi-axes 2 and 1, Major axis inclined at $\tan^{-1} 2$ to the axis of x.
5. Parabola ; axis $2x-y+1=0$, vertex $(-7/5, -9/5)$, focus $(-6/5, -7/5)$.
6. Hyperbola ; centre $(-1, 2)$, axes $2x+3y=3$ and $2x-y+4=0$.

7. Two parallel straight lines ; $x+3y+5=0$ and $x+3y-1=0$.
8. Ellipse ; centre $(-2, -2)$, Semi-axes $\sqrt{2}$ and $2/\sqrt{3}$, Major axis inclined at $\tan^{-1}(-4/3)$ to the axis of x.
9. Rectangular Hyperbola ; centre $(-1, 2)$, Semi-axes 1 each, transverse axis inclined at $\tan^{-1} 4/3$ to the axis of x.
10. Two parallel straight lines ;
    $x+2y+1=0$ and $x+2y+6=0$.
11. Two coincident straight lines.
12. An imaginary ellipse.
13. Two imaginary straight lines.
14. Two straight lines ; $5x+2y=3$ and $4x-3y=9$.
15. Two imaginary straight lines.
16. Hyperbola ; centre (1, 1), Semi-axes $\sqrt{5}$ and $\sqrt{5}/2$, Asymptotes $x-2y+1=0$ and $x+2y-3=0$; axes parallel to co-ordinate axes.

17. Ellipse ; axes $4x+3y-12=0$ and $3x-4y+12=0$, Semi-axes 3 and 2.

18. Parabola ; axes $x-y=0$, vertex $(a/4, a/4)$, focus $(a/2, a/2)$.

19. Ellipse ; axes $x+2y-2=0$ and $2x-y+1=0$, Semi-axes 3 and 3/2.

20. $e=2/\sqrt{3}$; centre $(-1/5, 3/5)$; axes $x+2y=1$ and $2x-y+1=0$.

21. Directrix $3x+2y+4=0$; $e=1/\sqrt{3}$; focus $(12/13, 8/13)$.

22. Centre $(3/2, 1/2)$; Foci $\left(\frac{3}{2}\pm\sqrt{2},\ \frac{1}{2}\pm\frac{\sqrt{2}}{3}\right)$.

23. Parabola ; axis $x+2y=0$, vertex $(0, 0)$, focus $(-3/5, 3/10)$ Directrix $4x-2y=3$.

24. Hyperbola ; Centre $(-1/5, 7/5)$, Semi-axes 2 and $4\sqrt{3}$; $e=\sqrt{13}/3$, asymptotes $8x-y+3=0$ and $4x+7y-9=0$.

25. Focus $(a/2, a/2)$; Directrix $x+y=0$.

Art. 10·12 (page 164)

1. $x^2-xy+y^2-2x-2y=0$; An ellipse.
2. $9x^2+25y^2-90x=0$.
3. $x^2\pm 2xy+y^2-3x-4=0$.
4. $x^2+y^2=25$.
5. $12x^2-12xy+11y^2-24x-52y=0$.

Art. 10·21 (page 166)

1. Two.

4. $8x^2+2hxy+3y^2-32x-18y+24=0$.

Art. 10·25 (page 169)

1. $(x-y-2)(x-y+1)+\lambda(2x+y+2)(4x-y-8)=0.$
   For parabolas $\lambda=0$ *or* 1.
   $x^2-6xy+3y^2-x+5y-2=0.$

2. $2x^2+4xy-3y^2-5x-5y+2=0.$

Art. 10·31 (page 171)

1. $\pm\dfrac{\alpha}{a}\pm\dfrac{\beta}{b}=1.$

Miscellaneous Exercises (pages 172–173)

1 $25(x-3)^2+9(y-5)^2+2hxy=225.$

2. $3x^2+5xy+y^2-y+2=0.$

3. $\dfrac{ax+hy+g}{hx+by+f}=\dfrac{a'x+h'y+g'}{h'x+b'y+f'}.$

6. A line parallel to the line joining the two given points and passing through the origin.

9. $7y^2-24xy+20x=0.$

10. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1+\dfrac{e^2}{b^2}(x-ae)^2=0.$

12. $x^2-2xy+y^2-14x-14y+1=0$; axis $x-y=0$; focus $(25/14,\ 25/14)$.

13. $a+b=a'+b'$ for those passing through the common centre ; $\dfrac{a-b}{h}=\dfrac{a'-b'}{h'}$ for those not passing through the common centre.

15. $9x^2-9xy-3y^2+2x=0.$

Art. 10·40 (page 174)

2. $\dfrac{2x^2}{3-\sqrt{5}}-\dfrac{2y^2}{\sqrt{5}-1}=1$

   $\dfrac{2x^2}{3+\sqrt{5}}+\dfrac{2y^2}{1+\sqrt{5}}=1.$